

SPDEs with α -stable Lévy noise: a random field approach

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Abstract

This article is dedicated to the study of an SPDE of the form

$$Lu(t, x) = \sigma(u(t, x))\dot{Z}(t, x) \quad t > 0, x \in \mathcal{O}$$

with zero initial conditions and Dirichlet boundary conditions, where σ is a Lipschitz function, L is a second-order pseudo-differential operator, \mathcal{O} is a bounded domain in \mathbb{R}^d , and \dot{Z} is an α -stable Lévy noise with $\alpha \in (0, 2)$, $\alpha \neq 1$ and possibly non-symmetric tails. To give a meaning to the concept of solution, we develop a theory of stochastic integration with respect to Z , by generalizing the method of [11] to higher dimensions and non-symmetric tails. The idea is to first solve the equation with “truncated” noise \dot{Z}_K (obtained by removing from Z the jumps which exceed a fixed value K), yielding a solution u_K , and then show that the solutions $u_L, L > K$ coincide on the event $t \leq \tau_K$, for some stopping times $\tau_K \uparrow \infty$ a.s. A similar idea was used in [22] in the setting of Hilbert-space valued processes. A major step is to show that the stochastic integral with respect to Z_K satisfies a p -the moment inequality, for $p \in (\alpha, 1)$ if $\alpha < 1$, and $p \in (\alpha, 2)$ if $\alpha > 1$. This inequality plays the same role as the Burkholder-Davis-Gundy inequality in the theory of integration with respect to continuous martingales.

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1 Introduction

Modeling phenomena which evolve in time or space-time and are subject to random perturbations is a fundamental problem in stochastic analysis. When these perturbations are known to exhibit an extreme behavior, as seen frequently in finance or environmental studies, a model relying on the Gaussian distribution is not appropriate. A suitable alternative could be a model based on a heavy-tailed distribution, like the stable distribution. In such a model, these perturbations are allowed to have extreme values with a probability which is significantly higher than in a Gaussian-based model.

In the present article, we introduce precisely such a model, given rigorously by a stochastic partial differential equation (SPDE) driven by a noise term which has a stable distribution over any space-time region, and has independent values over disjoint space-time regions (i.e. it is a Lévy noise). More precisely, we consider the SPDE:

$$Lu(t, x) = \sigma(u(t, x))\dot{Z}(t, x), \quad t > 0, x \in \mathcal{O} \quad (1)$$

with zero initial conditions and Dirichlet boundary conditions, where σ is a Lipschitz function, L is a second-order pseudo-differential operator on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$, and $\dot{Z}(t, x) = \frac{\partial^{d+1} Z}{\partial t \partial x_1 \dots \partial x_d}$ is the formal derivative of an α -stable Lévy noise with $\alpha \in (0, 2)$, $\alpha \neq 1$. The goal is to find sufficient conditions on the fundamental solution $G(t, x, y)$ of the equation $Lu = 0$, which will ensure the existence of a mild solution of equation (1). We say that a predictable process $u = \{u(t, x); t \geq 0, x \in \mathcal{O}\}$ is a **mild solution** of (1) if for any $t > 0, x \in \mathcal{O}$,

$$u(t, x) = \int_0^t \int_{\mathcal{O}} G(t, x, y) \sigma(u(s, y)) Z(ds, dy) \quad \text{a.s.} \quad (2)$$

We assume that $G(t, x, y)$ is a function in t , which excludes from our analysis the case of the wave equation with $d \geq 3$.

To explain the connections with other works, we describe briefly the construction of the noise (the details are given in Section 2 below). This construction is similar to that of a classical α -stable Lévy process, and is based on a Poisson random measure (PRM) N on $\mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$ of intensity $dt dx \nu_\alpha(dz)$, where

$$\nu_\alpha(dz) = [p\alpha z^{-\alpha-1} 1_{(0, \infty)}(z) + q\alpha(-z)^{-\alpha-1} 1_{(-\infty, 0)}(z)] dz \quad (3)$$

for some $p, q \geq 0$ with $p+q = 1$. More precisely, for any set $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$Z(B) = \int_{B \times \{|z| \leq 1\}} z \widehat{N}(ds, dx, dz) + \int_{B \times \{|z| > 1\}} z N(ds, dx, dz) - \mu |B|, \quad (4)$$

where $\widehat{N}(B \times \cdot) = N(B \times \cdot) - |B| \nu_\alpha(\cdot)$ is the compensated process and μ is a constant (specified by Lemma 2.3 below). Here, $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ is the class of bounded Borel sets in $\mathbb{R}_+ \times \mathbb{R}^d$ and $|B|$ is the Lebesgue measure of B .

As the term on the right-hand side of (2) is a stochastic integral with respect to Z , such an integral should be constructed first. Due to the poor integrability properties of the measure ν_α , this cannot be done directly from (4), using integration with respect to N and \widehat{N} . Our construction of the integral is an extension to random fields of the construction provided by Giné and Marcus in [11] in the case of an α -stable Lévy process $\{Z(t)\}_{t \in [0,1]}$. Unlike these authors, we do not assume that the measure ν_α is symmetric.

Since any Lévy noise is related to a PRM, in a broad sense, one could say that this problem originates in Itô's papers [12] and [13] regarding the stochastic integral with respect to a Poisson noise. SPDEs driven by a compensated PRM were considered for the first time in [14], using the approach based on Hilbert-space-valued solutions. This study was motivated by an application to neurophysiology leading to the cable equation. In the case of the heat equation, a similar problem was considered in [1], [26] and [3] using the approach based on random-field solutions. One of the results of [26] shows that the heat equation:

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \Delta u(t, x) + \int_U f(t, x, u(t, x); z) \widehat{N}(t, x, dz) + g(t, x, u(t, x))$$

has a unique solution in the space of predictable processes u satisfying $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E|u(t, x)|^p < \infty$, for any $p \in (1 + 2/d, 2]$. In this equation, \widehat{N} is the compensated process corresponding to a PRM N on $\mathbb{R}_+ \times \mathbb{R}^d \times U$ of intensity $dt dx \nu(dz)$, for an arbitrary σ -finite measure space $(U, \mathcal{B}(U), \nu)$ with measure ν satisfying $\int_U |z|^p \nu(dz) < \infty$. Because of this later condition, this result cannot be used in our case with $U = \mathbb{R} \setminus \{0\}$ and $\nu = \nu_\alpha$. For similar reasons, the results of [3] also do not cover the case of an α -stable noise. However, in the case $\alpha > 1$, we will be able to exploit successfully some ideas of [26] for treating the equation with “truncated” noise Z_K , obtained by removing from Z the jumps exceeding a value K (see Section 5.2 below).

The heat equation with the same type of noise as in the present article was examined in [16] and [18] in the cases $\alpha < 1$, respectively $\alpha > 1$, assuming that the noise has only positive jumps (i.e. $q = 0$). The methods used by these authors are different from those presented here, since they investigate the more difficult case of a non-Lipschitz function $\sigma(u) = u^\delta$ with $\delta > 0$. In [16], Mueller removes the atoms of Z of mass smaller than 2^{-n} and solves the equation driven by the noise obtained in this way; here we remove the atoms of Z of mass larger than K and solve the resulting equation. In [18], Mytnik uses a martingale problem approach and gives the existence of a pair (u, Z) which satisfies the equation (the so-called “weak solution”), whereas in the present article we obtain the existence of a solution u for a *given* noise Z (the so-called “strong solution”). In particular, when $\alpha > 1$ and $\delta = 1/\alpha$, the existence of a “weak solution” of the heat equation with α -stable Lévy noise is obtained in [18] under the condition

$$\alpha < 1 + \frac{2}{d} \tag{5}$$

which we encounter here as well. It is interesting to note that (5) is the necessary and sufficient condition for the existence of the density of the super-Brownian motion with “ $\alpha - 1$ ”-stable branching (see [7]). Reference [17] examines the heat equation with multiplicative noise (i.e. $\sigma(u) = u$), driven by an α -stable Lévy noise Z which does not depend on time.

To conclude the literature review, we should point out that there are many references related to stochastic differential equations with α -stable Lévy noise, using the approach based on Hilbert-space valued solutions. We refer the reader to Section 12.5 of the monograph [22], and to [21], [2], [15], [23] for a sample of relevant references. See also the survey article [20] for an approach based on the white noise theory for Lévy processes.

This article is organized as follows.

- In Section 2, we review the construction of the α -stable Lévy noise Z , and we show that this can be viewed as an independently scattered random measure with jointly α -stable distributions.
- In Section 3, we consider the linear equation (1) (with $\sigma(u) = 1$) and we identify the necessary and sufficient condition for the existence of the solution. This condition is verified in the case of some examples.

- Section 4 contains the construction of the stochastic integral with respect to the α -stable noise Z , for $\alpha \in (0, 2)$. The main effort is dedicated to proving a maximal inequality for the tail of the integral process, when the integrand is a simple process. This extends the construction of [11] to the case random fields and non-symmetric measure ν_α .
- In Section 5, we introduce the process Z_K obtained by removing from Z the jumps exceeding a fixed value K , and we develop a theory of integration with respect to this process. For this, we need to treat separately the cases $\alpha < 1$ and $\alpha > 1$. In both cases, we obtain a p -th moment inequality for the integral process for $p \in (\alpha, 1)$ if $\alpha < 1$, and $p \in (\alpha, 2)$ if $\alpha > 1$. This inequality plays the same role as the Burkholder-Davis-Gundy inequality in the theory of integration with respect to continuous martingales.
- In Section 6 we prove the main result about the existence of the mild solution of equation (1). For this, we first solve the equation with “truncated” noise Z_K using a Picard iteration scheme, yielding a solution u_K . We then introduce a sequence $(\tau_K)_{K \geq 1}$ of stopping times with $\tau_K \uparrow \infty$ a.s. and we show that the solutions $u_L, L > K$ coincide on the event $t \leq \tau_K$. For the definition of the stopping times τ_K , we need again to consider separately the cases $\alpha < 1$ and $\alpha > 1$.
- Appendix A contains some results about the tail of a non-symmetric stable random variable, and the tail of an infinite sum of random variables. Appendix B gives an estimate for the Green function associated to the fractional power of the Laplacian. Appendix C gives a local property of the stochastic integral with respect to Z (or Z_K).

2 Definition of the noise

In this section we review the construction of the α -stable Lévy noise on $\mathbb{R}_+ \times \mathbb{R}^d$ and investigate some of its properties.

Let $N = \sum_{i \geq 1} \delta_{(T_i, X_i, Z_i)}$ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$, defined on a probability space (Ω, \mathcal{F}, P) , with intensity measure $dt dx \nu_\alpha(dz)$, where ν_α is given by (3). Let $(\varepsilon_j)_{j \geq 0}$ be a sequence of positive real numbers such that $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ and $1 = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \dots$. Let

$$\Gamma_j = \{z \in \mathbb{R}; \varepsilon_j < |z| \leq \varepsilon_{j-1}\}, \quad j \geq 1 \quad \text{and} \quad \Gamma_0 = \{z \in \mathbb{R}; |z| > 1\}.$$

For any set $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we define

$$L_j(B) = \int_{B \times \Gamma_j} z N(dt, dx, dz) = \sum_{(T_i, X_i) \in B} Z_i 1_{\{Z_i \in \Gamma_j\}}, \quad j \geq 0.$$

Remark 2.1 The variable $L_0(B)$ is finite since the sum above contains finitely many terms. To see this, we use the “one-point uncompactification” as explained in Section 6.1.3 of [25], i.e. we view N as a point process on the space $\mathbb{R}_+ \times \mathbb{R}^d \times (\overline{\mathbb{R}} \setminus \{0\})$. Since the set Γ_0 is relatively compact in $\overline{\mathbb{R}} \setminus \{0\}$, and the point process N has finitely many points in any relatively compact set, $N(B \times \Gamma_0) = \text{card}\{i \geq 1; (T_i, X_i, J_i) \in B \times \Gamma_0\} < \infty$.

For any $j \geq 0$, the variable $L_j(B)$ has a compound Poisson distribution with jump intensity measure $|B| \cdot \nu_\alpha|_{\Gamma_j}$, i.e.

$$E[e^{iuL_j(B)}] = \exp \left\{ |B| \int_{\Gamma_j} (e^{iuz} - 1) \nu_\alpha(dz) \right\}, \quad u \in \mathbb{R}. \quad (6)$$

It follows that $E(L_j(B)) = |B| \int_{\Gamma_j} z \nu_\alpha(dz)$ and $\text{Var}(L_j(B)) = |B| \int_{\Gamma_j} z^2 \nu_\alpha(dz)$ for any $j \geq 0$. Hence $\text{Var}(L_j(B)) < \infty$ for any $j \geq 1$ and $\text{Var}(L_0(B)) = \infty$. If $\alpha > 1$, then $E(L_0(B))$ is finite. Define

$$Y(B) = \sum_{j \geq 1} [L_j(B) - E(L_j(B))] + L_0(B). \quad (7)$$

This sum converges a.s. by Kolmogorov's criterion since $\{L_j(B) - E(L_j(B))\}_{j \geq 1}$ are independent zero-mean random variables with $\sum_{j \geq 1} \text{Var}(L_j(B)) < \infty$.

From (6) and (7), it follows that $Y(B)$ is an infinitely divisible random variable with characteristic function:

$$E(e^{iuY(B)}) = \exp \left\{ |B| \int_{\mathbb{R}} (e^{iuz} - 1 - iuz 1_{\{|z| \leq 1\}}) \nu_\alpha(dz) \right\}, \quad u \in \mathbb{R}. \quad (8)$$

Hence $E(Y(B)) = |B| \int_{\mathbb{R}} z 1_{\{|z| > 1\}} \nu_\alpha(dz)$ and $\text{Var}(Y(B)) = |B| \int_{\mathbb{R}} z^2 \nu_\alpha(dz)$.

Lemma 2.2 *The family $\{Y(B); B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ defined by (7) is an independently scattered random measure, i.e.*

(a) *for any disjoint sets B_1, \dots, B_n in $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, $Y(B_1), \dots, Y(B_n)$ are independent ;*

(b) *for any sequence $(B_n)_{n \geq 1}$ of disjoint sets in $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\bigcup_{n \geq 1} B_n$ is bounded, $Y(\bigcup_{n \geq 1} B_n) = \sum_{n \geq 1} Y(B_n)$ a.s.*

Proof: (a) Note that for any function $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ with compact support K , we can define the random variable $Y(\varphi) = \sum_{j \geq 1} [L_j(\varphi) - E(L_j(\varphi))] + L_0(\varphi)$ where $L_j(\varphi) = \int_{K \times \Gamma_j} \varphi(t, x) z N(dt, dx, dz)$. For any $u \in \mathbb{R}$, we have:

$$E(e^{iuY(\varphi)}) = \exp \left\{ \int_{\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}} (e^{iuz\varphi(t,x)} - 1 - iuz\varphi(t,x)1_{\{|z| \leq 1\}}) dt dx \nu_\alpha(dz) \right\}. \quad (9)$$

For any disjoint sets B_1, \dots, B_n and for any $u_1, \dots, u_n \in \mathbb{R}$, we have:

$$\begin{aligned} E[\exp(i \sum_{k=1}^n u_k Y(B_k))] &= E[\exp(iY(\sum_{k=1}^n u_k 1_{B_k}))] \\ &= \exp \left\{ \int_{\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}} (e^{iz \sum_{k=1}^n u_k 1_{B_k}(t,x)} - 1 - iz 1_{\{|z| \leq 1\}} \sum_{k=1}^n u_k 1_{B_k}(t,x)) dt dx \nu_\alpha(dz) \right\} \\ &= \exp \left\{ \sum_{k=1}^n |B_k| \int_{\mathbb{R}} (e^{iu_k z} - 1 - iu_k z 1_{\{|z| \leq 1\}}) \nu_\alpha(dz) \right\} \\ &= \prod_{k=1}^n E[\exp(iu_k Y(B_k))], \end{aligned} \quad (10)$$

using (9) with $\varphi = \sum_{k=1}^n u_k 1_{B_k}$ for the second equality, and (6) for the last equality. This proves that $Y(B_1), \dots, Y(B_n)$ are independent.

(b) Let $S_n = \sum_{k=1}^n Y(B_k)$ and $S = Y(B)$, where $B = \bigcup_{n \geq 1} B_n$. By Lévy's equivalence theorem, $(S_n)_{n \geq 1}$ converges a.s. if and only if it converges in distribution. By (10), with $u_i = u$ for all $i = 1, \dots, k$, we have:

$$E(e^{iuS_n}) = \exp \left\{ \left| \bigcup_{k=1}^n B_k \right| \int_{\mathbb{R}} (e^{iuz} - 1 - iuz 1_{\{|z| \leq 1\}}) \nu_\alpha(dz) \right\}.$$

This clearly converges to $E(e^{iuS}) = \exp \left\{ |B| \int_{\mathbb{R}} (e^{iuz} - 1 - iuz 1_{\{|z| \leq 1\}}) \nu_\alpha(dz) \right\}$, and hence $(S_n)_{n \geq 1}$ converges in distribution to S . \square

Recall that a random variable X has an α -stable distribution with parameters $\alpha \in (0, 2)$, $\sigma \in [0, \infty)$, $\beta \in [-1, 1]$, $\mu \in \mathbb{R}$ if for any $u \in \mathbb{R}$,

$$\begin{aligned} E(e^{iuX}) &= \exp \left\{ -|u|^\alpha \sigma^\alpha \left(1 - i \operatorname{sgn}(u) \beta \tan \frac{\pi \alpha}{2} \right) + iu\mu \right\} \quad \text{if } \alpha \neq 1, \text{ or} \\ E(e^{iuX}) &= \exp \left\{ -|u| \sigma \left(1 + i \operatorname{sgn}(u) \beta \frac{2}{\pi} \ln |u| \right) + iu\mu \right\} \quad \text{if } \alpha = 1 \end{aligned}$$

(see Definition 1.1.6 of [27]). We denote this distribution by $S_\alpha(\sigma, \beta, \mu)$.

Lemma 2.3 $Y(B)$ has a $S_\alpha(\sigma|B|^{1/\alpha}, \beta, \mu|B|)$ distribution with $\beta = p - q$,

$$\sigma^\alpha = \int_0^\infty \frac{\sin x}{x^\alpha} dx = \begin{cases} \frac{\Gamma(2-\alpha)}{1-\alpha} \cos \frac{\pi\alpha}{2} & \text{if } \alpha \neq 1 \\ \frac{\pi}{2} & \text{if } \alpha = 1 \end{cases}, \quad \mu = \begin{cases} \beta \frac{\alpha}{\alpha-1} & \text{if } \alpha \neq 1 \\ \beta c_0 & \text{if } \alpha = 1 \end{cases}$$

and $c_0 = \int_0^\infty (\sin z - z1_{\{z \leq 1\}})z^{-2}dz$. If $\alpha > 1$, then $E(Y(B)) = \mu|B|$.

Proof: We first express the characteristic function (8) of $Y(B)$ in Feller's canonical form (see Section XVII.2 of [9]):

$$E(e^{iuY(B)}) = \exp \left\{ iub|B| + |B| \int_{\mathbb{R}} \frac{e^{iuz} - 1 - iu \sin z}{z^2} M_\alpha(dz) \right\}$$

with $M_\alpha(dz) = z^2 \nu_\alpha(dz)$ and $b = \int_{\mathbb{R}} (\sin z - z1_{\{|z| \leq 1\}}) \nu_\alpha(dz)$. Then the result follows from the calculations done in Example XVII.3.(g) of [9]. \square

From Lemma 2.2 and Lemma 2.3, it follows that

$$Z = \{Z(B) = Y(B) - \mu|B|; B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$$

is an α -stable random measure, in the sense of Definition 3.3.1 of [27], with control measure $m(B) = \sigma^\alpha|B|$ and constant skewness intensity β . In particular, $Z(B)$ has a $S_\alpha(\sigma|B|^{1/\alpha}, \beta, 0)$ distribution.

We say that Z is an **α -stable Lévy noise**. Coming back to the original construction (7) of $Y(B)$ and noticing that

$$\mu|B| = -|B| \int_{\mathbb{R}} z1_{\{|z| \leq 1\}} \nu_\alpha(dz) = - \sum_{j \geq 1} E(L_j(B)) \quad \text{if } \alpha < 1, \text{ and}$$

$$\mu|B| = |B| \int_{\mathbb{R}} z1_{\{|z| > 1\}} \nu_\alpha(dz) = E(L_0(B)) \quad \text{if } \alpha > 1,$$

it follows that $Z(B)$ can be represented as:

$$Z(B) = \sum_{j \geq 0} L_j(B) = \int_{B \times (\mathbb{R} \setminus \{0\})} zN(dt, dx, dz) \quad \text{if } \alpha < 1, \quad (11)$$

$$Z(B) = \sum_{j \geq 0} [L_j(B) - E(L_j(B))] = \int_{B \times (\mathbb{R} \setminus \{0\})} z\hat{N}(dt, dx, dz) \quad \text{if } \alpha > 1. \quad (12)$$

Here \hat{N} is the compensated Poisson measure associated to N , i.e. $\hat{N}(A) = N(A) - E(N(A))$ for any relatively compact set A in $\mathbb{R}_+ \times \mathbb{R}^d \times (\overline{\mathbb{R}} \setminus \{0\})$.

In the case $\alpha = 1$, we will assume that $p = q$ so that ν_α is symmetric around 0, $E(L_j(B)) = 0$ for all $j \geq 1$, and $Z(B)$ admits the same representation as in the case $\alpha < 1$.

3 The linear equation

As a preliminary investigation, we consider first equation (1) with $\sigma = 1$:

$$Lu(t, x) = \dot{Z}(t, x), \quad t > 0, x \in \mathcal{O} \quad (13)$$

with zero initial conditions and Dirichlet boundary conditions. In this section \mathcal{O} is a bounded domain in \mathbb{R}^d or $\mathcal{O} = \mathbb{R}^d$.

By definition, the process $\{u(t, x); t \geq 0, x \in \mathcal{O}\}$ given by:

$$u(t, x) = \int_0^t \int_{\mathcal{O}} G(t-s, x, y) Z(ds, dy) \quad (14)$$

is a mild solution of (13), provided that the stochastic integral on the right-hand side of (14) is well-defined.

We define now the stochastic integral of a deterministic function φ :

$$Z(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t, x) Z(dt, dx).$$

If $\varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$, this can be defined by approximation with simple functions, as explained in Section 3.4 of [27]. The process $\{Z(\varphi); \varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)\}$ has jointly α -stable finite dimensional distributions. In particular, each $Z(\varphi)$ has a $S_\alpha(\sigma_\varphi, \beta, 0)$ -distribution with scale parameter:

$$\sigma_\varphi = \sigma \left(\int_0^\infty \int_{\mathbb{R}^d} |\varphi(t, x)|^\alpha dx dt \right)^{1/\alpha}.$$

More generally, a measurable function $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ is integrable with respect to Z if there exists a sequence $(\varphi_n)_{n \geq 1}$ of simple functions such that $\varphi_n \rightarrow \varphi$ a.e., and for any $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, the sequence $\{Z(\varphi_n 1_B)\}_n$ converges in probability (see [24]).

The next results shows that condition $\varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$ is also necessary for the integrability of φ with respect to Z . Due to Lemma 2.2, this follows immediately from the general theory of stochastic integration with respect to independently scattered random measures developed in [24].

Lemma 3.1 *A deterministic function φ is integrable with respect to Z if and only if $\varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$.*

Proof: We write the characteristic function of $Z(B)$ in the form used in [24]:

$$E(e^{iuZ(B)}) = \exp \left\{ \int_B \left[iua + \int_{\mathbb{R}} (e^{iuz} - 1 - iu\tau(z))\nu_{\alpha}(dz) \right] dt dx \right\}$$

with $a = \beta - \mu$, $\tau(z) = z$ if $|z| \leq 1$ and $\tau(z) = \text{sgn}(z)$ if $|z| > 1$. By Theorem 2.7 of [24], φ is integrable with respect to Z if and only if

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} |U(\varphi(t, x))| dt dx < \infty \quad \text{and} \quad \int_{\mathbb{R}_+ \times \mathbb{R}^d} V(\varphi(t, x)) dt dx < \infty$$

where $U(y) = ay + \int_{\mathbb{R}} (\tau(yz) - y\tau(z))\nu_{\alpha}(dz)$ and $V(y) = \int_{\mathbb{R}} (1 \wedge |yz|^2)\nu_{\alpha}(dz)$. Direct calculations show that in our case, $U(y) = -\frac{\beta}{\alpha-1}y^{\alpha}$ if $\alpha \neq 1$, $U(y) = 0$ if $\alpha = 1$, and $V(y) = \frac{2}{2-\alpha}y^{\alpha}$. \square

The following result follows immediately from (14) and Lemma 3.1.

Proposition 3.2 *Equation (13) has a mild solution if and only if for any $t > 0, x \in \mathcal{O}$*

$$I_{\alpha}(t) = \int_0^t \int_{\mathcal{O}} G(s, x, y)^{\alpha} dy ds < \infty. \quad (15)$$

In this case, $\{u(t, x); t \geq 0, x \in \mathcal{O}\}$ has jointly α -stable finite-dimensional distributions. In particular, $u(t, x)$ has a $S_{\alpha}(\sigma I_{\alpha}(t)^{1/\alpha}, \beta, 0)$ distribution.

Condition (15) can be easily verified in the case of several examples.

Example 3.3 (*Heat equation*) Let $L = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$. Assume first that $\mathcal{O} = \mathbb{R}^d$. Then $G(t, x, y) = \overline{G}(t, x - y)$, where

$$\overline{G}(t, x) = \frac{1}{(2\pi t)^{d/2}} \exp \left(-\frac{|x|^2}{2t} \right), \quad (16)$$

and condition (15) is equivalent to (5). In this case, $I_{\alpha}(t) = c_{\alpha, d} t^{d(1-\alpha)/2+1}$. If \mathcal{O} is a bounded domain in \mathbb{R}^d , then $G(t, x, y) \leq \overline{G}(t, x - y)$ (see p. 74 of [17]) and condition (15) is implied by (5).

Example 3.4 (*Parabolic equations*) Let $L = \frac{\partial}{\partial t} - \mathcal{L}$ where

$$\mathcal{L}f(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x) \quad (17)$$

is the generator of a Markov process with values in \mathbb{R}^d , without jumps (a diffusion). Assume that \mathcal{O} is a bounded domain in \mathbb{R}^d or $\mathcal{O} = \mathbb{R}^d$. By Aronson estimate (see e.g. Theorem 2.6 of [22]), under some assumptions on the coefficients a_{ij}, b_i , there exist some constants $c_1, c_2 > 0$ such that

$$G(t, x, y) \leq c_1 t^{-d/2} \exp\left(-\frac{|x-y|^2}{c_2 t}\right) \quad (18)$$

for all $t > 0$ and $x, y \in \mathcal{O}$. In this case, condition (15) is implied by (5).

Example 3.5 (*Heat equation with fractional power of the Laplacian*) Let $L = \frac{\partial}{\partial t} + (-\Delta)^\gamma$ for some $\gamma > 0$. Assume that \mathcal{O} is a bounded domain in \mathbb{R}^d or $\mathcal{O} = \mathbb{R}^d$. Then (see e.g. Appendix B.5 of [22])

$$G(t, x, y) = \int_0^\infty \mathcal{G}(s, x, y) g_{t,\gamma}(s) ds = \int_0^\infty \mathcal{G}(t^{1/\gamma} s, x, y) g_{1,\gamma}(s) ds, \quad (19)$$

where $\mathcal{G}(t, x, y)$ is the fundamental solution of $\frac{\partial u}{\partial t} - \Delta u = 0$ on \mathcal{O} and $g_{t,\gamma}$ is the density of the measure $\mu_{t,\gamma}$, $(\mu_{t,\gamma})_{t \geq 0}$ being a convolution semigroup of measures on $[0, \infty)$ whose Laplace transform is given by:

$$\int_0^\infty e^{-us} g_{t,\gamma}(s) ds = \exp(-tu^\gamma), \quad \forall u > 0.$$

Note that if $\gamma < 1$, $g_{t,\gamma}$ is the density of S_t , where $(S_t)_{t \geq 0}$ is a γ -stable subordinator with Lévy measure $\rho_\gamma(dx) = \frac{\gamma}{\Gamma(1-\gamma)} x^{-\gamma-1} 1_{(0,\infty)}(x) dx$.

Assume first that $\mathcal{O} = \mathbb{R}^d$. Then $G(t, x, y) = \overline{G}(t, x - y)$, where

$$\overline{G}(t, x) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-t|\xi|^{2\gamma}} d\xi. \quad (20)$$

If $\gamma < 1$, then $\overline{G}(t, \cdot)$ is the density of X_t , $(X_t)_{t \geq 0}$ being a symmetric (2γ) -stable Lévy process with values in \mathbb{R}^d defined by $X_t = W_{S_t}$, with $(W_t)_{t \geq 0}$ a Brownian motion in \mathbb{R}^d with variance 2. By Lemma B.1 (Appendix B), if $\alpha > 1$, then (15) holds if and only if

$$\alpha < 1 + \frac{d}{2\gamma}. \quad (21)$$

If \mathcal{O} is a bounded domain in \mathbb{R}^d , then $G(t, x, y) \leq \overline{G}(t, x - y)$ (by Lemma 2.1 of [16]). In this case, if $\alpha > 1$, then (15) is implied by (21).

Example 3.6 (*Cable equation in \mathbb{R}*) Let $Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u$ and $\mathcal{O} = \mathbb{R}$. Then $G(t, x, y) = \overline{G}(t, x - y)$, where

$$\overline{G}(t, x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t} - t\right),$$

and condition (15) holds for any $\alpha \in (0, 2)$.

Example 3.7 (*Wave equation in \mathbb{R}^d with $d = 1, 2$*) Let $L = \frac{\partial^2}{\partial t^2} - \Delta$ and $\mathcal{O} = \mathbb{R}^d$ with $d = 1$ or $d = 2$. Then $G(t, x, y) = \overline{G}(t, x - y)$, where

$$\overline{G}(t, x) = \frac{1}{2} 1_{\{|x| < t\}} \quad \text{if } d = 1$$

$$\overline{G}(t, x) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x| < t\}} \quad \text{if } d = 2.$$

Condition (15) holds for any $\alpha \in (0, 2)$. In this case, $I_\alpha(t) = 2^{-\alpha} t^2$ if $d = 1$ and $I_\alpha(t) = \frac{(2\pi)^{1-\alpha}}{(2-\alpha)(3-\alpha)} t^{3-\alpha}$ if $d = 2$.

4 Stochastic integration

In this section we construct a stochastic integral with respect Z by generalizing the ideas of [11] to the case of random fields. Unlike these authors, we do not assume that $Z(B)$ has a symmetric distribution, unless $\alpha = 1$.

Let $\mathcal{F}_t = \mathcal{F}_t^N \vee \mathcal{N}$ where \mathcal{N} is the σ -field of negligible sets in (Ω, \mathcal{F}, P) and \mathcal{F}_t^N is the σ -field generated by $N([0, s] \times A \times \Gamma)$ for all $s \in [0, t]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and for all Borel sets $\Gamma \subset \mathbb{R} \setminus \{0\}$ bounded away from 0. Note that $\mathcal{F}_t^Z \subset \mathcal{F}_t^N$ where \mathcal{F}_t^Z is the σ -field generated by $Z([0, s] \times A)$, $s \in [0, t]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$.

A process $X = \{X(t, x)\}_{t \geq 0, x \in \mathbb{R}^d}$ is called *elementary* if it of the form

$$X(t, x) = 1_{(a, b]}(t) 1_A(x) Y \tag{22}$$

where $0 \leq a < b$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and Y is \mathcal{F}_a -measurable. A *simple process* is a linear combination of elementary processes. Note that any simple process X can be written as:

$$X(t, x) = 1_{\{0\}}(t) Y_0(x) + \sum_{i=0}^{N-1} 1_{(t_i, t_{i+1}]}(t) Y_i(x) \tag{23}$$

with $0 = t_0 < t_1 < \dots < t_N < \infty$ and $Y_i(x) = \sum_{j=1}^{m_i} 1_{A_{ij}}(x)Y_{ij}$, where $(Y_{ij})_{j=1,\dots,m_i}$ are \mathcal{F}_{t_i} -measurable and $(A_{ij})_{j=1,\dots,m_j}$ are disjoint sets in $\mathcal{B}_b(\mathbb{R}^d)$. Without loss of generality, we assume that $Y_0 = 0$.

We denote by \mathcal{P} the *predictable σ -field* on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$, i.e. the σ -field generated by all simple processes. We say that a process $X = \{X(t, x)\}_{t \geq 0, x \in \mathbb{R}^d}$ is *predictable* if the map $(\omega, t, x) \mapsto X(\omega, t, x)$ is \mathcal{P} -measurable.

Remark 4.1 One can show that the predictable σ -field \mathcal{P} is the σ -field generated by the class \mathcal{C} of processes X such that $t \mapsto X(\omega, t, x)$ is left-continuous for any $\omega \in \Omega, x \in \mathbb{R}^d$ and $(\omega, x) \mapsto X(\omega, t, x)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$ -measurable for any $t > 0$.

Let \mathcal{L}_α be the class of all predictable processes X such that

$$\|X\|_{\alpha, T, B}^\alpha := E \int_0^T \int_B |X(t, x)|^\alpha dx dt < \infty,$$

for all $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$. Note that \mathcal{L}_α is a linear space.

Let $(E_k)_{k \geq 1}$ be an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_k E_k = \mathbb{R}^d$. We define

$$\begin{aligned} \|X\|_\alpha &= \sum_{k \geq 1} \frac{1 \wedge \|X\|_{\alpha, k, E_k}^\alpha}{2^k} \quad \text{if } \alpha > 1, \\ \|X\|_\alpha^\alpha &= \sum_{k \geq 1} \frac{1 \wedge \|X\|_{\alpha, k, E_k}^\alpha}{2^k} \quad \text{if } \alpha \leq 1. \end{aligned}$$

We identify two processes X and Y for which $\|X - Y\|_\alpha = 0$, i.e. $X = Y$ ν -a.e., where $\nu = P dtdx$. In particular, we identify two processes X and Y if X is a modification of Y , i.e. $X(t, x) = Y(t, x)$ a.s. for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

The space \mathcal{L}_α becomes a metric space endowed with the metric d_α :

$$d_\alpha(X, Y) = \|X - Y\|_\alpha \text{ if } \alpha > 1, \quad d_\alpha(X, Y) = \|X - Y\|_\alpha^\alpha \text{ if } \alpha \leq 1.$$

This follows using Minkowski's inequality if $\alpha > 1$, and the inequality $|a + b|^\alpha \leq |a|^\alpha + |b|^\alpha$ if $\alpha \leq 1$.

The following result can be proved similarly to Proposition 2.3 of [29].

Proposition 4.2 *For any $X \in \mathcal{L}_\alpha$ there exists a sequence $(X_n)_{n \geq 1}$ of bounded simple processes such that $\|X_n - X\|_\alpha \rightarrow 0$ as $n \rightarrow \infty$.*

By Proposition 5.7 of [25], the process $\{Z(t, B) = Z([0, t] \times B); t \geq 0\}$ has a càdlàg modification, for any $B \in \mathcal{B}_b(\mathbb{R}^d)$. We work with these modifications. If X is a simple process given by (23), we define

$$I(X)(t, B) = \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} Y_{ij} Z((t_i \wedge t, t_{i+1} \wedge t] \times (A_{ij} \cap B)). \quad (24)$$

Note that for any $B \in \mathcal{B}_b(\mathbb{R}^d)$, $I(X)(t, B)$ is \mathcal{F}_t -measurable for any $t \geq 0$, and $\{I(X)(t, B)\}_{t \geq 0}$ is càdlàg. We write

$$I(X)(t, B) = \int_0^t \int_B X(s, x) Z(ds, dx).$$

The following result will be used for the construction of the integral. This result generalizes Lemma 3.3 of [11] to the case of random fields and non-symmetric measures ν_α .

Theorem 4.3 *If X is a bounded simple process then*

$$\sup_{\lambda > 0} \lambda^\alpha P\left(\sup_{t \in [0, T]} |I(X)(t, B)| > \lambda\right) \leq c_\alpha E \int_0^T \int_B |X(t, x)|^\alpha dx dt, \quad (25)$$

for any $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$, where c_α is a constant depending only on α .

Proof: Suppose that X is of the form (23). Since $\{I(X)(t, B)\}_{t \in [0, T]}$ is càdlàg, it is separable. Without loss of generality, we assume that its separating set D can be written as $D = \cup_n F_n$ where $(F_n)_n$ is an increasing sequence of finite sets containing the points $(t_k)_{k=0, \dots, N}$. Hence,

$$P\left(\sup_{t \in [0, T]} |I(X)(t, B)| > \lambda\right) = \lim_{n \rightarrow \infty} P\left(\max_{t \in F_n} |I(X)(t, B)| > \lambda\right). \quad (26)$$

Fix $n \geq 1$. Denote by $0 = s_0 < s_1 < \dots < s_m = T$ the points of the set F_n . Say $t_k = s_{i_k}$ for some $0 = i_0 < i_1 < \dots < i_N$. Then each interval $(t_k, t_{k+1}]$ can be written as the union of some intervals of the form $(s_i, s_{i+1}]$:

$$(t_k, t_{k+1}] = \bigcup_{i \in I_k} (s_i, s_{i+1}] \quad (27)$$

where $I_k = \{i; i_k \leq i < i_{k+1}\}$. By (24), for any $k = 0, \dots, N-1$ and $i \in I_k$,

$$I(X)(s_{i+1}, B) - I(X)(s_i, B) = \sum_{j=1}^{m_k} Y_{kj} Z((s_i, s_{i+1}] \times (A_{kj} \cap B)).$$

For any $i \in I_k$, let $N_i = m_k$, and for any $j = 1, \dots, N_i$, define $\beta_{ij} = Y_{kj}$, $H_{ij} = A_{kj}$ and $Z_{ij} = Z((s_i, s_{i+1}] \times (H_{ij} \cap B))$. With this notation, we have:

$$I(X)(s_{i+1}, B) - I(X)(s_i, B) = \sum_{j=1}^{N_i} \beta_{ij} Z_{ij} \quad \text{for all } i = 0, \dots, m.$$

Consequently, for any $l = 1, \dots, m$

$$I(X)(s_l, B) = \sum_{i=0}^{l-1} (I(X)(s_{i+1}, B) - I(X)(s_i, B)) = \sum_{i=0}^{l-1} \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}. \quad (28)$$

Using (26) and (28), it is enough to prove that for any $\lambda > 0$,

$$P(\max_{l=0, \dots, m-1} |\sum_{i=0}^l \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}| > \lambda) \leq c_\alpha \lambda^{-\alpha} E \int_0^T \int_B |X(s, x)|^\alpha dx ds. \quad (29)$$

First, note that

$$E \int_0^T \int_B |X(s, x)|^\alpha dx ds = \sum_{i=0}^{m-1} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E |\beta_{ij}|^\alpha |H_{ij} \cap B|.$$

This follows from the definition (23) of X and (27), since $X(t, x) = \sum_{i=0}^{N-1} \sum_{i \in I_k} 1_{(s_i, s_{i+1}]}(t) \sum_{j=1}^{N_i} \beta_{ij} 1_{H_{ij}}(x)$.

We now prove (29). Let $W_i = \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}$. For the event on the left-hand side, we consider its intersection with the event $\{\max_{0 \leq i \leq m-1} |W_i| > \lambda\}$ and its complement. Hence the probability of this event can be bounded by

$$\sum_{i=0}^{m-1} P(|W_i| > \lambda) + P(\max_{0 \leq l \leq m-1} |\sum_{i=0}^l W_i 1_{\{|W_i| \leq \lambda\}}| > \lambda) =: I + II.$$

We treat separately the two terms.

For the first term, we note that $\bar{\beta}_i = (\beta_{ij})_{1 \leq j \leq N_i}$ is \mathcal{F}_{s_i} -measurable and $\bar{Z}_i = (Z_{ij})_{1 \leq j \leq N_i}$ is independent of \mathcal{F}_{s_i} . By Fubini's theorem

$$I = \sum_{i=0}^{m-1} \int_{\mathbb{R}^{N_i}} P(|\sum_{j=1}^{N_i} x_j Z_{ij}| > \lambda) P_{\bar{\beta}_i}(d\bar{x}),$$

where $\bar{x} = (x_j)_{1 \leq j \leq N_i}$ and $P_{\bar{\beta}_i}$ is the law of $\bar{\beta}_i$.

We examine the tail of $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$ for a fixed $\bar{x} \in \mathbb{R}^{N_i}$. By Lemma 2.3, Z_{ij} has a $S_\alpha(\sigma(s_{i+1} - s_i)^{1/\alpha} |H_{ij} \cap B|^{1/\alpha}, \beta, 0)$ distribution. Since the sets $(H_{ij})_{1 \leq j \leq N_i}$ are disjoint, the variables $(Z_{ij})_{1 \leq j \leq N_i}$ are independent. Using elementary properties of the stable distribution (Properties 1.2.1 and 1.2.3 of [27]), it follows that U_i has a $S_\alpha(\sigma_i, \beta_i^*, 0)$ distribution with parameters:

$$\sigma_i^\alpha = \sigma^\alpha(s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|$$

$$\beta_i^* = \frac{\beta}{\sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|} \sum_{j=1}^{N_i} \text{sgn}(x_j) |x_j|^\alpha |H_{ij} \cap B|.$$

By Lemma A.1 (Appendix A), there exists a constant $c_\alpha^* > 0$ such that

$$P(|U_i| > \lambda) \leq c_\alpha^* \lambda^{-\alpha} \sigma^\alpha(s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B| \quad (30)$$

for any $\lambda > 0$. Hence

$$I \leq c_\alpha^* \lambda^{-\alpha} \sigma^\alpha \sum_{i=0}^{m-1} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B| = c_\alpha^* \lambda^{-\alpha} \sigma^\alpha E \int_0^T \int_B |X(s, x)|^\alpha dx ds.$$

We now treat *II*. We consider three cases. For the first two cases we deviate from the original argument of [11] since we do not require that $\beta = 0$.

Case 1. $\alpha < 1$. Note that

$$II \leq P(\max_{0 \leq l \leq m-1} M_l > \lambda) \quad (31)$$

where $\{M_l = \sum_{i=0}^l |W_i| 1_{\{|W_i| \leq \lambda\}}, \mathcal{F}_{s_{l+1}}; 0 \leq l \leq m-1\}$ is a submartingale. By the submartingale maximal inequality (Theorem 35.3 of [5]),

$$P(\max_{0 \leq l \leq m-1} M_l > \lambda) \leq \frac{1}{\lambda} E(M_{m-1}) = \frac{1}{\lambda} \sum_{i=0}^{m-1} E(|W_i| 1_{\{|W_i| \leq \lambda\}}). \quad (32)$$

Using the independence between $\bar{\beta}_i$ and \bar{Z}_i it follows that

$$E[|W_i|1_{|W_i|\leq\lambda}] = \int_{\mathbb{R}^{N_i}} E\left[\left|\sum_{j=1}^{N_i} x_j Z_{ij}\right|1_{\left|\sum_{j=1}^{N_i} x_j Z_{ij}\right|\leq\lambda}\right] P_{\bar{\beta}_i}(d\bar{x})$$

Let $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$. Using (30) and Remark A.2 (Appendix A), we get:

$$E[|U_i|1_{|U_i|\leq\lambda}] \leq c_\alpha^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{1-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|.$$

Hence

$$E[|W_i|1_{|W_i|\leq\lambda}] \leq c_\alpha^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{1-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B|. \quad (33)$$

From (31), (32) and (33), it follows that:

$$II \leq c_\alpha^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{-\alpha} E \int_0^T \int_B |X(s, x)|^\alpha dx ds.$$

Case 2. $\alpha > 1$. We have

$$II \leq P\left(\max_{0 \leq l \leq m-1} \left|\sum_{i=0}^l X_i\right| > \lambda/2\right) + P\left(\max_{0 \leq l \leq m-1} Y_l > \lambda/2\right) =: II' + II'',$$

where $X_i = W_i 1_{\{|W_i|\leq\lambda\}} - E[W_i 1_{\{|W_i|\leq\lambda\}} | \mathcal{F}_{s_i}]$ and $Y_i = |E[W_i 1_{\{|W_i|\leq\lambda\}} | \mathcal{F}_{s_i}]|$.

We first treat the term II' . Note that $\{M_l = \sum_{i=0}^l X_i, \mathcal{F}_{s_{l+1}}; 0 \leq l \leq m-1\}$ is a zero-mean square integrable martingale, and

$$II' = P\left(\max_{0 \leq l \leq m-1} |M_l| > \lambda/2\right) \leq \frac{4}{\lambda^2} \sum_{i=0}^{m-1} E(X_i^2) \leq \frac{4}{\lambda^2} \sum_{i=0}^{m-1} E[W_i^2 1_{\{|W_i|\leq\lambda\}}].$$

Let $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$. Using (30) and Remark A.2 (Appendix A), we get:

$$E[U_i^2 1_{\{|U_i|\leq\lambda\}}] \leq 2c_\alpha^* \sigma^\alpha \frac{1}{2-\alpha} \lambda^{2-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|.$$

As in Case 1, we obtain that:

$$E[W_i^2 1_{\{|W_i| \leq \lambda\}}] \leq c_\alpha^* \sigma^\alpha \frac{2}{2-\alpha} \lambda^{2-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B|, \quad (34)$$

and hence

$$II' \leq 8c_\alpha^* \sigma^\alpha \frac{1}{2-\alpha} \lambda^{-\alpha} E \int_0^T \int_B |X(s, x)|^\alpha dx ds.$$

We now treat II'' . Note that $\{N_l = \sum_{i=0}^l Y_i, \mathcal{F}_{s_{l+1}}; 0 \leq l \leq m-1\}$ is a semimartingale and hence, by the submartingale inequality,

$$II'' \leq \frac{2}{\lambda} E(N_{m-1}) = \frac{2}{\lambda} \sum_{i=0}^{m-1} E(Y_i).$$

To evaluate $E(Y_i)$, we note that for almost all $\omega \in \Omega$,

$$E[W_i 1_{\{|W_i| \leq \lambda\}} | \mathcal{F}_{s_i}](\omega) = E\left[\sum_{j=1}^{N_i} \beta_{ij}(\omega) Z_{ij} 1_{\{|\sum_{j=1}^{N_i} \beta_{ij}(\omega) Z_{ij}| \leq \lambda\}}\right], \quad (35)$$

due to the independence between $\bar{\beta}_i$ and \bar{Z}_i . We let $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$ with $x_j = \beta_{ij}(\omega)$. Since $\alpha > 1$, $E(U_i) = 0$. Using (30) and Remark A.2, we obtain

$$\begin{aligned} |E[U_i 1_{\{|U_i| \leq \lambda\}}]| &= |E[U_i 1_{\{|U_i| > \lambda\}}]| \leq E[|U_i| 1_{\{|U_i| > \lambda\}}] \\ &\leq c_\alpha^* \sigma^\alpha \frac{\alpha}{\alpha-1} \lambda^{1-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|. \end{aligned}$$

Hence, $E(Y_i) \leq c_\alpha^* \sigma^\alpha \frac{\alpha}{\alpha-1} \lambda^{1-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B|$ and

$$II'' \leq c_\alpha^* \sigma^\alpha \frac{2\alpha}{\alpha-1} \lambda^{-\alpha} E \int_0^T \int_B |X(t, x)|^\alpha dx dt.$$

Case 3. $\alpha = 1$. In this case we assume that $\beta = 0$. Hence $U_i = \sum_{j=1}^{N_i} x_j Z_{ij}$ has a symmetric distribution for any $\bar{x} \in \mathbb{R}^{N_i}$. Using (35), it follows that $E[W_i 1_{\{|W_i| \leq \lambda\}} | \mathcal{F}_{s_i}] = 0$ a.s. for all $i = 0, \dots, m-1$. Hence, $\{M_l = \sum_{i=0}^l W_i 1_{\{|W_i| \leq \lambda\}}, \mathcal{F}_{s_{l+1}}; 0 \leq l \leq m-1\}$ is a zero-mean square integrable martingale. By the martingale maximal inequality,

$$II \leq \frac{1}{\lambda^2} E[M_{m-1}^2] = \frac{1}{\lambda^2} \sum_{i=0}^{m-1} E[W_i^2 1_{\{|W_i| \leq \lambda\}}].$$

The result follows using (34). \square

We now proceed to the construction of the stochastic integral. If $Y = \{Y(t)\}_{t \geq 0}$ is a jointly measurable random process, we define:

$$\|Y\|_{\alpha, T}^\alpha = \sup_{\lambda > 0} \lambda^\alpha P\left(\sup_{t \in [0, T]} |Y(t)| > \lambda\right).$$

Let $X \in \mathcal{L}_\alpha$ be arbitrary. By Proposition 4.2, there exists a sequence $(X_n)_{n \geq 1}$ of simple functions such that $\|X_n - X\|_\alpha \rightarrow 0$ as $n \rightarrow \infty$. Let $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$ be fixed. By linearity of the integral and Theorem 4.3,

$$\|I(X_n)(\cdot, B) - I(X_m)(\cdot, B)\|_{\alpha, T}^\alpha \leq c_\alpha \|X_n - X_m\|_{\alpha, T, B}^\alpha \rightarrow 0 \quad (36)$$

as $n, m \rightarrow \infty$. In particular, the sequence $\{I(X_n)(\cdot, B)\}_n$ is Cauchy in probability in the space $D[0, T]$ equipped with the sup-norm. Therefore, there exists a random element $Y(\cdot, B)$ in $D[0, T]$ such that for any $\lambda > 0$,

$$P\left(\sup_{t \in [0, T]} |I(X_n)(t, B) - Y(t, B)| > \lambda\right) \rightarrow 0.$$

Moreover, there exists a subsequence $(n_k)_k$ such that

$$\sup_{t \in [0, T]} |I(X_{n_k})(t, B) - Y(t, B)| \rightarrow 0 \quad \text{a.s.}$$

as $k \rightarrow \infty$. Hence $Y(t, B)$ is \mathcal{F}_t -measurable for any $t \in [0, T]$. The process $Y(\cdot, B)$ does not depend on the sequence $(X_n)_n$ and can be extended to a càdlàg process on $[0, \infty)$, which is unique up to indistinguishability. We denote this extension by $I(X)(\cdot, B)$ and we write

$$I(X)(t, B) = \int_0^t \int_B X(s, x) Z(ds, dx).$$

If A and B are disjoint sets in $\mathcal{B}_b(\mathbb{R}^d)$, then

$$I(X)(t, A \cup B) = I(X)(t, A) + I(X)(t, B) \quad \text{a.s.} \quad (37)$$

Lemma 4.4 *Inequality (25) holds for any $X \in \mathcal{L}_\alpha$.*

Proof: Let $(X_n)_n$ be a sequence of simple functions such that $\|X_n - X\|_\alpha \rightarrow 0$. For fixed B , we denote $I(X) = I(X)(\cdot, B)$. We let $\|\cdot\|_\infty$ be the sup norm on $D[0, T]$. For any $\varepsilon > 0$, we have:

$$P(\|I(X)\|_\infty > \lambda) \leq P(\|I(X) - I(X_n)\|_\infty > \lambda\varepsilon) + P(\|I(X_n)\|_\infty > \lambda(1 - \varepsilon)).$$

Multiplying by λ^α , and using Theorem 4.3, we obtain:

$$\sup_{\lambda>0} \lambda^\alpha P(\|I(X)\|_\infty > \lambda) \leq \varepsilon^{-\alpha} \sup_{\lambda>0} \lambda^\alpha P(\|I(X) - I(X_n)\|_\infty > \lambda) + (1-\varepsilon)^{-\alpha} c_\alpha \|X_n\|_{\alpha,T,B}^\alpha.$$

Let $n \rightarrow \infty$. Using (36) one can prove that $\sup_{\lambda>0} \lambda^\alpha P(\|I(X_n) - I(X)\|_\infty > \lambda) \rightarrow 0$. We obtain that $\sup_{\lambda>0} \lambda^\alpha P(\|I(X)\|_\infty > \lambda) \leq (1-\varepsilon)^{-\alpha} c_\alpha \|X\|_{\alpha,T,B}^\alpha$. The conclusion follows letting $\varepsilon \rightarrow 0$. \square

For an arbitrary Borel set $\mathcal{O} \subset \mathbb{R}^d$ (possibly $\mathcal{O} = \mathbb{R}^d$), we assume in addition, that $X \in \mathcal{L}_\alpha$ satisfies the condition:

$$E \int_0^T \int_{\mathcal{O}} |X(t, x)|^\alpha dx dt < \infty \quad \text{for all } T > 0. \quad (38)$$

Then we can define $I(X)(\cdot, \mathcal{O})$ as follows. Let $\mathcal{O}_k = \mathcal{O} \cap E_k$ where $(E_k)_k$ is an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_k E_k = \mathbb{R}^d$. By (37), Lemma 4.4 and (38),

$$\sup_{\lambda>0} \lambda^\alpha P(\sup_{t \leq T} |I(X)(t, \mathcal{O}_k) - I(X)(t, \mathcal{O}_l)| > \lambda) \leq c_\alpha E \int_0^T \int_{\mathcal{O}_k \setminus \mathcal{O}_l} |X(t, x)|^\alpha dx dt \rightarrow 0$$

as $k, l \rightarrow \infty$. This shows that $\{I(X)(\cdot, \mathcal{O}_k)\}_k$ is a Cauchy sequence in probability in the space $D[0, T]$ equipped with the sup norm. We denote by $I(X)(\cdot, \mathcal{O})$ its limit. As above, this process can be extended to $[0, \infty)$ and $I(X)(t, \mathcal{O})$ is \mathcal{F}_t -measurable for any $t > 0$. We denote

$$I(X)(t, \mathcal{O}) = \int_0^t \int_{\mathcal{O}} X(s, x) Z(ds, dx).$$

Similarly to Lemma 4.4, one can prove that for any $X \in \mathcal{L}_\alpha$ satisfying (38),

$$\sup_{\lambda>0} \lambda^\alpha P(\sup_{t \leq T} |I(X)(t, \mathcal{O})| > \lambda) \leq c_\alpha E \int_0^T \int_{\mathcal{O}} |X(t, x)|^\alpha dx dt.$$

5 The truncated noise

For the study of non-linear equations, we need to develop a theory of stochastic integration with respect to another process Z_K which is defined by removing from Z the jumps whose modulus exceed a fixed value $K > 0$. More precisely, for any $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we define

$$Z_K(B) = \int_{B \times \{0 < |z| \leq K\}} z N(ds, dx, dz) \quad \text{if } \alpha \leq 1 \quad (39)$$

$$Z_K(B) = \int_{B \times \{0 < |z| \leq K\}} z \widehat{N}(ds, dx, dz) \quad \text{if } \alpha > 1. \quad (40)$$

We treat separately the cases $\alpha \leq 1$ and $\alpha > 1$.

5.1 The case $\alpha \leq 1$

Note that $\{Z_K(B); B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is an independently scattered random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with characteristic function given by:

$$E(e^{iuZ_K(B)}) = \exp \left\{ |B| \int_{|z| \leq K} (e^{iuz} - 1) \nu_\alpha(dz) \right\} \quad \forall u \in \mathbb{R}.$$

We first examine the tail of $Z_K(B)$.

Lemma 5.1 *For any set $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$,*

$$\sup_{\lambda > 0} \lambda^\alpha P(|Z_K(B)| > \lambda) \leq r_\alpha |B| \quad (41)$$

where $r_\alpha > 0$ a constant depending only on α (given by Lemma A.3).

Proof: This follows from Example 3.7 of [11]. We denote by $\nu_{\alpha,K}$ the restriction of ν_α to $\{z \in \mathbb{R}; 0 < |z| \leq K\}$. Note that

$$\nu_{\alpha,K}(\{z \in \mathbb{R}; |z| > t\}) = \begin{cases} t^{-\alpha} - K^{-\alpha} & \text{if } 0 < t \leq K \\ 0 & \text{if } t > K \end{cases}$$

and hence $\sup_{t > 0} t^\alpha \nu_{\alpha,K}(\{z \in \mathbb{R}; |z| > t\}) = 1$. Next we observe that we do not need to assume that the measure $\nu_{\alpha,K}$ is symmetric since we use a modified version of Lemma 2.1 of [10] given by Lemma A.3 (Appendix A). \square

In fact, since the tail of $\nu_{\alpha,K}$ vanishes if $t > K$, we can obtain another estimate for the tail of $Z_K(B)$ which, together with (41), will allow us to control its p -th moment for $p \in (\alpha, 1)$. This new estimate is given below.

Lemma 5.2 *If $\alpha < 1$, then*

$$P(|Z_K(B)| > u) \leq \frac{\alpha}{1-\alpha} K^{1-\alpha} |B| u^{-1} \quad \text{for all } u > K.$$

If $\alpha = 1$, then $P(|Z_K(B)| > u) \leq K |B| u^{-2}$ for all $u > K$.

Proof: We use the same idea as in Example 3.7 of [11]. For each $k \geq 1$, let $Z_{k,K}(B)$ be a random variable with characteristic function:

$$E(e^{iuZ_{k,K}(B)}) = \exp \left\{ |B| \int_{\{k^{-1} < |z| \leq K\}} (e^{iuz} - 1) \nu_\alpha(dz) \right\}.$$

Since $\{Z_{k,K}(B)\}_k$ converges in distribution to $Z_K(B)$, it suffices to prove the lemma for $Z_{k,K}(B)$. Let μ_k be the restriction of ν_α to $\{z; k^{-1} < |z| \leq K\}$. Since μ_k is finite, $Z_{k,K}(B)$ has a compound Poisson distribution with

$$P(|Z_{k,K}(B)| > u) = e^{-|B|\mu_k(\mathbb{R})} \sum_{n \geq 0} \frac{|B|^n}{n!} \mu_k^{*n}(\{z; |z| > u\}). \quad (42)$$

where μ_k^{*n} denotes the n -fold convolution. Note that

$$\mu_k^{*n}(\{z; |z| > u\}) = [\mu_k(\mathbb{R})]^n P(|\sum_{i=1}^n \eta_i| > u),$$

where $(\eta_i)_{i \geq 1}$ are i.i.d. random variables with law $\mu_k/\mu_k(\mathbb{R})$.

Assume first that $\alpha < 1$. To compute $P(|\sum_{i=1}^n \eta_i| > u)$ we consider the intersection with the event $\{\max_{1 \leq i \leq n} |\eta_i| > u\}$ and its complement. Note that $P(|\eta_i| > u) = 0$ for any $u > K$. Using this fact and Markov's inequality, we obtain that for any $u > K$,

$$P(|\sum_{i=1}^n \eta_i| > u) \leq P(|\sum_{i=1}^n \eta_i 1_{\{|\eta_i| \leq u\}}| > u) \leq \frac{1}{u} \sum_{i=1}^n E(|\eta_i| 1_{\{|\eta_i| \leq u\}}).$$

Note that $P(|\eta_i| > s) \leq (s^{-\alpha} - K^{-\alpha})/\mu_k(\mathbb{R})$ if $s \leq K$. Hence, for any $u > K$

$$E(|\eta_i| 1_{\{|\eta_i| \leq u\}}) \leq \int_0^u P(|\eta_i| > s) ds = \int_0^K P(|\eta_i| > s) ds \leq \frac{1}{\mu_k(\mathbb{R})} \frac{\alpha}{1-\alpha} K^{1-\alpha}.$$

Combining all these facts, we get: for any $u > K$

$$\mu_k^{*n}(\{z; |z| > u\}) \leq [\mu_k(\mathbb{R})]^{n-1} \frac{\alpha}{1-\alpha} K^{1-\alpha} n u^{-1},$$

and the conclusion follows from (42).

Assume now that $\alpha = 1$. In this case, $E(\eta_i 1_{\{|\eta_i| \leq u\}}) = 0$ since η_i has a symmetric distribution. Using Chebyshev's inequality this time, we obtain:

$$P(|\sum_{i=1}^n \eta_i| > u) \leq P(|\sum_{i=1}^n \eta_i 1_{\{|\eta_i| \leq u\}}| > u) \leq \frac{1}{u^2} \sum_{i=1}^n E(\eta_i^2 1_{\{|\eta_i| \leq u\}}).$$

The result follows as above using the fact that for any $u > K$,

$$E(\eta_i^2 1_{\{|\eta_i| \leq u\}}) \leq 2 \int_0^u sP(|\eta_i| > s)ds = 2 \int_0^K sP(|\eta_i| > s)ds \leq \frac{1}{\mu_k(\mathbb{R})}K.$$

□

Lemma 5.3 *If $\alpha < 1$ then*

$$E|Z_K(B)|^p \leq C_{\alpha,p}K^{p-\alpha}|B| \quad \text{for any } p \in (\alpha, 1),$$

where $C_{\alpha,p}$ is a constant depending on α and p . If $\alpha = 1$, then

$$E|Z_K(B)|^p \leq C_pK^{p-1}|B| \quad \text{for any } p \in (1, 2),$$

where C_p is a constant depending on p .

Proof: Note that

$$E|Z_K(B)|^p = \int_0^\infty P(|Z_K(B)|^p > t)dt = p \int_0^\infty P(|Z_K(B)| > u)u^{p-1}du.$$

We consider separately the integrals for $u \leq K$ and $u > K$. For the first integral we use (41):

$$\int_0^K P(|Z_K(B)| > u)u^{p-1}du \leq r_\alpha|B| \int_0^K u^{-\alpha+p-1}du = r_\alpha|B| \frac{1}{p-\alpha}K^{p-\alpha}.$$

For the second one we use Lemma 5.2: if $\alpha < 1$ then

$$\int_K^\infty P(|Z_K(B)| > u)u^{p-1}du \leq \frac{\alpha}{1-\alpha}K^{1-\alpha}|B| \int_K^\infty u^{p-2}du = \frac{\alpha}{(1-\alpha)(1-p)}|B|K^{p-\alpha},$$

and if $\alpha = 1$, then

$$\int_K^\infty P(|Z_K(B)| > u)u^{p-1}du \leq K|B| \int_K^\infty u^{p-3}du = |B| \frac{1}{2-p}K^{p-1}.$$

□

We now proceed to the construction of the stochastic integral with respect to Z_K . For this, we use the same method as for Z . Note that $\mathcal{F}_t^{Z_K} \subset \mathcal{F}_t$, where $\mathcal{F}_t^{Z_K}$ is the σ -field generated by $Z_K([0, s] \times A)$ for all $s \in [0, t]$ and

$A \in \mathcal{B}_b(\mathbb{R}^d)$. For any $B \in \mathcal{B}_b(\mathbb{R}^d)$, we will work with a càdlàg modification of the Lévy process $\{Z_K(t, B) = Z_K([0, t] \times B); t \geq 0\}$.

If X is a simple process given by (23), we define

$$I_K(X)(t, B) = \int_0^t \int_B X(s, x) Z_K(ds, dx)$$

by the same formula (24) with Z replaced by Z_K . The following result shows that $I_K(X)(t, B)$ has the same tail behavior as $I(X)(t, B)$.

Proposition 5.4 *If X is a bounded simple process then*

$$\sup_{\lambda > 0} \lambda^\alpha P\left(\sup_{t \in [0, T]} |I_K(X)(t, B)| > \lambda\right) \leq d_\alpha E \int_0^T \int_B |X(t, x)|^\alpha dx dt, \quad (43)$$

for any $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$, where d_α is a constant depending only on α .

Proof: As in the proof of Theorem 4.3, it is enough to prove that

$$P\left(\max_{l=0, \dots, m-1} \left| \sum_{i=0}^l \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}^* \right| > \lambda\right) \leq d_\alpha \lambda^{-\alpha} \sum_{i=0}^{m-1} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E |\beta_{ij}|^\alpha |H_{ij} \cap B|,$$

where $Z_{ij}^* = Z_K((s_i, s_{i+1}] \times (H_{ij} \cap B))$. This reduces to showing that $U_i^* = \sum_{j=1}^{N_i} x_j Z_{ij}^*$ satisfies an inequality similar to (30) for any $\bar{x} \in \mathbb{R}^{N_i}$, i.e.

$$P(|U_i^*| > \lambda) \leq d_\alpha^* \lambda^{-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|, \quad (44)$$

for any $\lambda > 0$, for some $d_\alpha^* > 0$. We first examine the tail of Z_{ij}^* . By (41),

$$P(|Z_{ij}^*| > \lambda) \leq r_\alpha (s_{i+1} - s_i) K_{ij} \lambda^{-\alpha}.$$

where $K_{ij} = |H_{ij} \cap B|$. Letting $\eta_{ij} = K_{ij}^{-1/\alpha} Z_{ij}^*$, we obtain that for any $u > 0$,

$$P(|\eta_{ij}| > u) \leq r_\alpha (s_{i+1} - s_i) u^{-\alpha} \quad \forall j = 1, \dots, N_i.$$

By Lemma A.3 (Appendix A), it follows that for any $\lambda > 0$,

$$P\left(\left| \sum_{j=1}^{N_i} b_j \eta_{ij} \right| > \lambda\right) \leq r_\alpha^2 (s_{i+1} - s_i) \sum_{j=1}^{N_i} |b_j|^\alpha \lambda^{-\alpha},$$

for any sequence $(b_j)_{j=1,\dots,N_i}$ of real numbers. Inequality (44) (with $d_\alpha^* = r_\alpha^2$) follows by applying this to $b_j = x_j K_{ij}^{1/\alpha}$. \square

In view of the previous result and Proposition 4.2, for any process $X \in \mathcal{L}_\alpha$ we can construct the integral

$$I_K(X)(t, B) = \int_0^t \int_B X(s, x) Z_K(ds, dx)$$

in the same manner as $I(X)(t, B)$, and this integral satisfies (43). If in addition the process $X \in \mathcal{L}_\alpha$ satisfies (38), then we can define the integral $I_K(X)(t, \mathcal{O})$ for an arbitrary Borel set $\mathcal{O} \subset \mathbb{R}^d$ (possibly $\mathcal{O} = \mathbb{R}^d$). This integral will satisfy an inequality similar to (43) with B replaced by \mathcal{O} .

The appealing feature of $I_K(X)(t, B)$ is that we can control its moments, as shown by the next result.

Theorem 5.5 *If $\alpha < 1$, then for any $p \in (\alpha, 1)$ and for any $X \in \mathcal{L}_p$,*

$$E|I_K(X)(t, B)|^p \leq C_{\alpha,p} K^{p-\alpha} E \int_0^t \int_B |X(s, x)|^p dx ds, \quad (45)$$

for any $t > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$, where $C_{\alpha,p}$ is a constant depending on α, p . If $\mathcal{O} \subset \mathbb{R}^d$ is an arbitrary Borel set, and we assume in addition, that the process $X \in \mathcal{L}_p$ satisfies:

$$E \int_0^T \int_{\mathcal{O}} |X(s, x)|^p dx ds < \infty \quad \forall T > 0, \quad (46)$$

then inequality (45) holds with B replaced by \mathcal{O} .

Proof: *Step 1.* Suppose that X is an elementary process of the form (22). Then $I_K(X)(t, B) = Y Z_K(H)$ where $H = (t \wedge a, t \wedge b] \times (A \cap B)$. Note that $Z_K(H)$ is independent of \mathcal{F}_a . Hence, $Z_K(H)$ is independent of Y . Let P_Y denote the law of Y . By Fubini's theorem,

$$\begin{aligned} E|Y Z_K(H)|^p &= p \int_0^\infty P(|Y Z_K(H)| > u) u^{p-1} du \\ &= p \int_{\mathbb{R}} \left(\int_0^\infty P(|y Z_K(H)| > u) u^{p-1} du \right) P_Y(dy). \end{aligned}$$

We evaluate the inner integral. We split this integral into two parts, for $u \leq K|y|$, respectively $u > K|y|$. For the first integral, we use (41). For the second one, we use Lemma 5.2. Therefore, the inner integral is bounded by:

$$r_\alpha |y|^\alpha |H| \int_0^{K|y|} u^{-\alpha+p-1} du + \frac{\alpha}{1-\alpha} |y| K^{1-\alpha} |H| \int_{K|y|}^\infty u^{p-2} du = C'_{\alpha,p} K^{p-\alpha} |y|^p |H|$$

and

$$E|YZ_K(H)|^p \leq p C'_{\alpha,p} K^{p-\alpha} |H| E|Y|^p = C_{\alpha,p} K^{p-\alpha} E \int_0^t \int_B |X(s, x)|^p dx ds.$$

Step 2. Suppose now that X is a simple process of the form (23). Then $X(t, x) = \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} X_{ij}(t, x)$ where $X_{ij}(t, x) = 1_{(t_i, t_{i+1}]}(t) 1_{A_{ij}}(x) Y_{ij}$.

Using the linearity of the integral, the inequality $|a+b|^p \leq |a|^p + |b|^p$, and the result obtained in Step 1 for the elementary processes X_{ij} , we get:

$$\begin{aligned} E|I_K(X)(t, B)|^p &\leq E \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} |I_K(X_{ij})(t, B)|^p \leq \\ &C_{\alpha,p} K^{p-\alpha} E \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} \int_0^t \int_B |X_{ij}(s, x)|^p dx ds = C_{\alpha,p} K^{p-\alpha} E \int_0^t \int_B |X(s, x)|^p dx ds. \end{aligned}$$

Step 3. Let $X \in \mathcal{L}_p$ be arbitrary. By Proposition 4.2, there exists a sequence $(X_n)_n$ of bounded simple processes such that $\|X_n - X\|_p \rightarrow 0$. Since $\alpha < p$, it follows that $\|X_n - X\|_\alpha \rightarrow 0$. By the definition of $I_K(X)(t, B)$ there exists a subsequence $\{n_k\}_k$ such that $\{I_K(X_{n_k})(t, B)\}_k$ converges to $I_K(X)(t, B)$ a.s. Using Fatou's lemma and the result obtained in Step 2 (for the simple processes X_{n_k}), we get:

$$\begin{aligned} E|I_K(X)(t, B)|^p &\leq \liminf_{k \rightarrow \infty} E|I_K(X_{n_k})(t, B)|^p \\ &\leq C_{\alpha,p} K^{p-\alpha} \liminf_{k \rightarrow \infty} E \int_0^t \int_B |X_{n_k}(s, x)|^p dx ds \\ &= C_{\alpha,p} K^{p-\alpha} E \int_0^t \int_B |X(s, x)|^p dx ds. \end{aligned}$$

Step 4. Suppose that $X \in \mathcal{L}_p$ satisfies (46). Let $\mathcal{O}_k = \mathcal{O} \cap E_k$ where $(E_k)_k$ is an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_{k \geq 1} E_k = \mathbb{R}^d$.

By the definition of $I_K(X)(t, \mathcal{O})$, there exists a subsequence $(k_i)_i$ such that $\{I_K(X)(t, \mathcal{O}_{k_i})\}_i$ converges to $I_K(X)(t, \mathcal{O})$ a.s. Using Fatou's lemma, the result obtained in Step 3 (for $B = \mathcal{O}_{k_i}$) and the monotone convergence theorem, we get:

$$\begin{aligned} E|I_K(X)(t, \mathcal{O})|^p &\leq \liminf_{i \rightarrow \infty} E|I_K(X)(t, \mathcal{O}_{k_i})|^p \\ &\leq C_{\alpha,p} K^{p-\alpha} \liminf_{i \rightarrow \infty} E \int_0^t \int_{\mathcal{O}_{k_i}} |X(s, x)|^p dx ds \\ &= C_{\alpha,p} K^{p-\alpha} E \int_0^t \int_{\mathcal{O}} |X(s, x)|^p dx ds. \end{aligned}$$

□

Remark 5.6 Finding a similar moment inequality for the case $\alpha = 1$ and $p \in (1, 2)$ remains an open problem. The argument used in Step 2 above relies on the fact that $p < 1$. Unfortunately, we could not find another argument to cover the case $p > 1$.

5.2 The case $\alpha > 1$

In this case, the construction of the integral with respect to Z_K relies on an integral with respect to \hat{N} which exists in the literature. We recall briefly the definition of this integral. For more details, see Section 1.2.2 of [26], Section 24.2 of [28] or Section 8.7 of [22].

Let $\mathbb{E} = \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$ endowed with the measure $\mu(dx, dz) = dx \nu_\alpha(dz)$ and $\mathcal{B}_b(\mathbb{E})$ be the class of bounded Borel sets in \mathbb{E} . For a simple process $Y = \{Y(t, x, z); t \geq 0, (x, z) \in \mathbb{E}\}$, the integral $I^{\hat{N}}(Y)(t, B)$ is defined in the usual way, for any $t > 0, B \in \mathcal{B}_b(\mathbb{E})$. The process $I^{\hat{N}}(Y)(\cdot, B)$ is a (càdlàg) zero-mean square-integrable martingale with quadratic variation

$$[I^{\hat{N}}(Y)(\cdot, B)]_t = \int_0^t \int_B |Y(s, x, z)|^2 N(ds, dx, dz)$$

and predictable quadratic variation

$$\langle I^{\hat{N}}(Y)(\cdot, B) \rangle_t = \int_0^t \int_B |Y(s, x, z)|^2 \nu_\alpha(dz) dx ds.$$

By approximation, this integral can be extended to the class of all $\mathcal{P} \times \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable processes Y such that, for any $T > 0$ and $B \in \mathcal{B}_b(\mathbb{E})$

$$\|Y\|_{2,T,B}^2 := E \int_0^T \int_B |Y(s, x, z)|^2 \nu_\alpha(dz) dx ds < \infty.$$

The integral is a martingale with the same quadratic variations as above, and has the isometry property: $E|I^{\hat{N}}(Y)(t, B)|^2 = \|Y\|_{2,T,B}^2$. If in addition, $\|Y\|_{2,T,\mathbb{E}} < \infty$, then the integral can be extended to \mathbb{E} . By the Burkholder-Davis-Gundy inequality for discontinuous martingales, for any $p \geq 1$

$$E \sup_{t \leq T} |I^{\hat{N}}(Y)(t, \mathbb{E})|^p \leq C_p E[I^{\hat{N}}(Y)(\cdot, \mathbb{E})]_T^{p/2}. \quad (47)$$

The previous inequality is not suitable for our purposes. A more convenient inequality can be obtained for *another* stochastic integral, constructed for $p \in [1, 2]$ fixed, as suggested on page 293 of [26]. More precisely, one can show that for any bounded simple process Y ,

$$E \sup_{t \leq T} |I^{\hat{N}}(Y)(t, \mathbb{E})|^p \leq C_p E \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R} \setminus \{0\}} |Y(t, x, z)|^p \nu_\alpha(dz) dx dt =: [Y]_{p,T,\mathbb{E}}^p, \quad (48)$$

where C_p is the constant appearing in (47) (see Lemma 8.22 of [22]).

By the usual procedure, the integral can be extended to the class of all $\mathcal{P} \times \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable processes Y such that $[Y]_{p,T,\mathbb{E}} < \infty$. The integral is defined as an element in the space $L^p(\Omega; D[0, T])$ and will be denoted by

$$I^{\hat{N},p}(Y)(t, \mathbb{E}) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R} \setminus \{0\}} Y(s, x, z) \hat{N}(ds, dx, dz).$$

Its appealing feature is that it satisfies inequality (48).

From now on, we fix $p \in [1, 2]$. Based on (40), for any $B \in \mathcal{B}_b(\mathbb{R}^d)$, we let

$$I_K(X)(t, B) = \int_0^t \int_B X(s, x) Z_K(ds, dx) = \int_0^t \int_B \int_{\{|z| \leq K\}} X(s, x) z \hat{N}(ds, dx, dz),$$

for any predictable process $X = \{X(t, x); t \geq 0, x \in \mathbb{R}^d\}$ for which the rightmost integral is well-defined. Letting $Y(t, x, z) = X(t, x) z 1_{\{0 < |z| \leq K\}}$, we see that this is equivalent to saying that $p > \alpha$ and $X \in \mathcal{L}_p$. By (48),

$$E \sup_{t \leq T} |I_K(X)(t, B)|^p \leq C_{\alpha,p} K^{p-\alpha} E \int_0^T \int_B |X(s, x)|^p dx ds \quad (49)$$

where $C_{\alpha,p} = C_p \alpha / (p - \alpha)$. If in addition, the process $X \in \mathcal{L}_p$ satisfies (46) then (49) holds with B replaced by \mathcal{O} , for an arbitrary Borel set $\mathcal{O} \subset \mathbb{R}^d$.

Note that (49) is the counterpart of (45) for the case $\alpha > 1$. Together, these two inequalities will play a crucial role in Section 6.

The following table summarizes all the conditions:

	$\alpha < 1$	$\alpha > 1$
B is bounded	$X \in \mathcal{L}_\alpha$	$X \in \mathcal{L}_p$ for some $p \in (\alpha, 2]$
$B = \mathcal{O}$ is unbounded	$X \in \mathcal{L}_\alpha$ and X satisfies (38)	$X \in \mathcal{L}_p$ and X satisfies (46) for some $p \in (\alpha, 2]$

Table 1: Conditions for $I_K(X)(t, B)$ to be well-defined

6 The main result

In this section, we state and prove the main result regarding the existence of a mild solution of equation (1). For this result, \mathcal{O} is a bounded domain in \mathbb{R}^d . For any $t > 0$, we denote

$$J_p(t) = \sup_{x \in \mathcal{O}} \int_{\mathcal{O}} G(t, x, y)^p dy.$$

Theorem 6.1 *Let $\alpha \in (0, 2)$, $\alpha \neq 1$. Assume that for any $T > 0$,*

$$\lim_{h \rightarrow 0} \int_0^T \int_{\mathcal{O}} |G(t, x, y) - G(t+h, x, y)|^p dy dt = 0 \quad \forall x \in \mathcal{O}, \quad (50)$$

$$\lim_{|h| \rightarrow 0} \int_0^T \int_{\mathcal{O}} |G(t, x, y) - G(t, x+h, y)|^p dy dt = 0 \quad \forall x \in \mathcal{O}, \quad (51)$$

$$\int_0^T J_p(t) dt < \infty, \quad (52)$$

for some $p \in (\alpha, 1)$ if $\alpha < 1$, or for some $p \in (\alpha, 2]$ if $\alpha > 1$. Then equation (1) has a mild solution. Moreover, there exists a sequence $(\tau_K)_{K \geq 1}$ of stopping times with $\tau_K \uparrow \infty$ a.s. such that for any $T > 0$,

$$\sup_{(t,x) \in [0,T] \times \mathcal{O}} E(|u(t, x)|^p 1_{\{t \leq \tau_K\}}) < \infty.$$

Example 6.2 (*Heat equation*) Let $L = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$. Then $G(t, x, y) \leq \overline{G}(t, x - y)$ where $\overline{G}(t, x)$ is the fundamental solution of $Lu = 0$ on \mathbb{R}^d . Condition (52) holds if $p < 1 + 2/d$. If $\alpha < 1$, this condition holds for any $p \in (\alpha, 1)$. If $\alpha > 1$, this condition holds for any $p \in (\alpha, 1 + 2/d]$, as long as α satisfies (5). Conditions (50) and (51) hold by the continuity of the function G in t and x , by applying the dominated convergence theorem. To justify the application of this theorem, we use the trivial bound $(2\pi t)^{-dp/2}$ for both $G(t+h, x, y)^p$ and $G(t, x+h, y)^p$, which introduces the extra condition $dp < 2$. Unfortunately, we could not find another argument for proving these two conditions. (In the case of the heat equation on \mathbb{R}^d , Lemmas A.2 and A.3 of [26] estimate the integrals appearing in (51) and (50), with $p = 1$ in (50). These arguments rely on the structure of \overline{G} and cannot be used when \mathcal{O} is a bounded domain.)

Example 6.3 (*Parabolic equations*) Let $L = \frac{\partial}{\partial t} - \mathcal{L}$ where \mathcal{L} is given by (17). Assuming (18), we see that (52) holds if $p < 1 + 2/d$. The same comments as for the heat equation apply here as well. (Although in a different framework, a condition similar to (50) was probably used in the proof of Theorem 12.11 of [22] (page 217) for the claim $\lim_{s \rightarrow t} E|J_3(X)(s) - J_3(X)(t)|_{L^p(\mathcal{O})}^p = 0$. We could not see how to justify this claim, unless $dp < 2$.)

Example 6.4 (*Heat equation with fractional power of the Laplacian*) Let $L = \frac{\partial}{\partial t} + (-\Delta)^\gamma$ for some $\gamma > 0$. By Lemma B.23 of [22], if $\alpha > 1$, then condition (52) holds for any $p \in (\alpha, 1 + 2\gamma/d)$, provided that α satisfies (21). (This condition is the same as in Theorem 12.19 of [22], which examines the same equation using the approach based on Hilbert-space valued solution.) To verify condition (50) and (51), we use the continuity of G in t and x and apply the dominated convergence theorem. To justify the application of this theorem, we use the trivial bound $C_{d,\gamma} t^{-dp/(2\gamma)}$ for both $G(t+h, x, y)^p$ and $G(t, x+h, y)^p$, which introduces the extra condition $dp < 2\gamma$. This bound can be seen from (19), using the fact that $\mathcal{G}(t, x, y) \leq \overline{\mathcal{G}}(t, x - y)$ where \mathcal{G} and $\overline{\mathcal{G}}$ are the fundamental solutions of $\frac{\partial u}{\partial t} - \Delta u = 0$ on \mathcal{O} , respectively \mathbb{R}^d . (In the case of the same equation on \mathbb{R}^d , elementary estimates for the time and space increments of $\overline{\mathcal{G}}$ can be obtained directly from (20), as on p. 196 of [4]. These arguments cannot be used when \mathcal{O} is a bounded domain.)

The remaining part of this section is dedicated to the proof of Theorem 6.1. The idea is to solve first the equation with the truncated noise Z_K (yielding a mild solution u_K), and then identify a sequence $(\tau_K)_{K \geq 1}$ of stopping times with $\tau_K \uparrow \infty$ a.s. such that for any $t > 0$, $x \in \mathcal{O}$ and $L > K$,

$u_K(t, x) = u_L(t, x)$ a.s. on the event $\{t \leq \tau_K\}$. The final step is to show that process u defined by $u(t, x) = u_K(t, x)$ on $\{t \leq \tau_K\}$ is a mild solution of (1). A similar method can be found in Section 9.7 of [22] using an approach based on stochastic integration of operator-valued processes, with respect to Hilbert-space-valued processes, which is different from our approach.

Since σ is a Lipschitz function, there exists a constant $C_\sigma > 0$ such that:

$$|\sigma(u) - \sigma(v)| \leq C_\sigma |u - v|, \quad \forall u, v \in \mathbb{R}. \quad (53)$$

In particular, letting $D_\sigma = C_\sigma \vee |\sigma(0)|$, we have:

$$|\sigma(u)| \leq D_\sigma(1 + |u|), \quad \forall u \in \mathbb{R}. \quad (54)$$

For the proof of Theorem 6.1, we need a specific construction of the Poisson random measure N , taken from [21]. We review briefly this construction.

Let $(\mathcal{O}_k)_{k \geq 1}$ be a partition of \mathbb{R}^d with sets in $\mathcal{B}_b(\mathbb{R}^d)$ and $(U_j)_{j \geq 1}$ be a partition of $\mathbb{R} \setminus \{0\}$ such that $\nu_\alpha(U_j) < \infty$ for all $j \geq 1$. We may take $U_j = \Gamma_{j-1}$ for all $j \geq 1$. Let $(E_i^{j,k}, X_i^{j,k}, Z_i^{j,k})_{i,j,k \geq 1}$ be independent random variables defined on a probability space (Ω, \mathcal{F}, P) , such that

$$P(E_i^{j,k} > t) = e^{-\lambda_{j,k}t}, \quad P(X_i^{j,k} \in B) = \frac{|B \cap \mathcal{O}_k|}{|\mathcal{O}_k|}, \quad P(Z_i^{j,k} \in \Gamma) = \frac{|\Gamma \cap U_j|}{|U_j|},$$

where $\lambda_{j,k} = |\mathcal{O}_k| \nu_\alpha(U_j)$. Let $T_i^{j,k} = \sum_{l=1}^i E_l^{j,k}$ for all $i \geq 1$. Then

$$N = \sum_{i,j,k \geq 1} \delta_{(T_i^{j,k}, X_i^{j,k}, Z_i^{j,k})} \quad (55)$$

is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$ with intensity $dt dx \nu_\alpha(dz)$.

This section is organized as follows. In Section 6.1 we prove the existence of the solution of the equation with truncated noise Z_K . Sections 6.2 and 6.3 contain the proof of Theorem 6.1 when $\alpha < 1$, respectively $\alpha > 1$.

6.1 The equation with truncated noise

In this section, we fix $K > 0$ and we consider the equation:

$$Lu(t, x) = \sigma(u(t, x)) \dot{Z}_K(t, x) \quad t > 0, x \in \mathcal{O} \quad (56)$$

with zero initial conditions and Dirichlet boundary conditions. A mild solution of (56) is a predictable process u which satisfies (2) with Z replaced by Z_K . For the next result, \mathcal{O} can be a bounded domain in \mathbb{R}^d or $\mathcal{O} = \mathbb{R}^d$ (with no boundary conditions).

Theorem 6.5 *Under the assumptions of Theorem 6.1, equation (56) has a unique mild solution $u = \{u(t, x); t \geq 0, x \in \mathcal{O}\}$. For any $T > 0$,*

$$\sup_{(t,x) \in [0,T] \times \mathcal{O}} E|u(t, x)|^p < \infty, \quad (57)$$

and the map $(t, x) \mapsto u(t, x)$ is continuous from $[0, T] \times \mathcal{O}$ into $L^p(\Omega)$.

Proof: We use the same argument as in the proof of Theorem 13 of [6], based on a Picard iteration scheme. We define $u_0(t, x) = 0$ and

$$u_{n+1}(t, x) = \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u_n(s, y)) Z_K(ds, dy)$$

for any $n \geq 0$. We prove by induction on $n \geq 0$ that: (i) $u_n(t, x)$ is well-defined; (ii) $K_n(t) := \sup_{(t,x) \in [0,T] \times \mathcal{O}} E|u_n(t, x)|^p < \infty$ for any $T > 0$; (iii) $u_n(t, x)$ is \mathcal{F}_t -measurable for any $t > 0$ and $x \in \mathcal{O}$; (iv) the map $(t, x) \mapsto u_n(t, x)$ is continuous from $[0, T] \times \mathcal{O}$ into $L^p(\Omega)$ for any $T > 0$.

The statement is trivial for $n = 0$. For the induction step, assume that the statement is true for n . By an extension to random fields of Theorem 30, Chapter IV of [8], u_n has a jointly measurable modification. Since this modification is $(\mathcal{F}_t)_t$ -adapted, (in the sense of (iii)), it has a predictable modification (using an extension of Proposition 3.21 of [22] to random fields). We work with this modification, that we call also u_n .

We prove that (i)-(iv) hold for u_{n+1} . To show (i), it suffices to prove that $X_n \in \mathcal{L}_p$, where $X_n(s, y) = 1_{[0,t]}(s)G(t-s, x, y)\sigma(u_n(s, y))$. By (54) and (52),

$$E \int_0^t \int_{\mathcal{O}} |X_n(s, y)|^p dy ds \leq D_{\sigma}^p 2^{p-1} (1 + K_n(t)) \int_0^t J_p(t-s) ds < \infty.$$

In addition, if $\mathcal{O} = \mathbb{R}^d$, we have to prove that X_n satisfies (38) if $\alpha < 1$, or (46) if $\alpha > 1$ (see Table 1). If $\alpha < 1$, this follows as above, since $\alpha < p$ and hence $\sup_{(t,x) \in [0,T] \times \mathcal{O}} E|u(t, x)|^{\alpha} < \infty$; the argument for $\alpha > 1$ is similar. Combined with the moment inequality (45) (or (49)), this proves (ii), since:

$$E|u_{n+1}(t, x)|^p \leq C_{\alpha,p} K^{p-\alpha} D_{\sigma}^p 2^{p-1} (1 + K_n(t)) \int_0^t J_p(t-s) ds, \quad (58)$$

for any $x \in \mathcal{O}$. Property (iii) follows by the construction of the integral I_K .

To prove (iv), we first show the right continuity in t . Let $h > 0$. Writing the interval $[0, t + h]$ as the union of $[0, t]$ and $(t, t + h]$, we obtain that $E|u_{n+1}(t + h, x) - u_{n+1}(t, x)|^p \leq 2^{p-1}(I_1(h) + I_2(h))$, where

$$\begin{aligned} I_1(h) &= E \left| \int_0^t \int_{\mathcal{O}} (G(t + h - s, x, y) - G(t - s, x, y)) \sigma(u_n(s, y)) Z_K(ds, dy) \right|^p \\ I_2(h) &= E \left| \int_t^{t+h} \int_{\mathcal{O}} G(t + h - s, x, y) \sigma(u_n(s, y)) Z_K(ds, dy) \right|^p. \end{aligned}$$

Using again (54) and the moment inequality (45) (or (49)), we obtain:

$$\begin{aligned} I_1(h) &\leq D_{\sigma}^p 2^{p-1} (1 + K_n(t)) \int_0^t \int_{\mathcal{O}} |G(s + h, x, y) - G(s, x, y)|^p dy ds \\ I_2(h) &\leq D_{\sigma}^p 2^{p-1} (1 + K_n(t)) \int_0^h \int_{\mathcal{O}} G(s, x, y)^p dy ds \end{aligned}$$

It follows that both $I_1(h)$ and $I_2(h)$ converge to 0 as $h \rightarrow 0$, using (50) for $I_1(h)$, respectively the Dominated Convergence Theorem and (52) for $I_2(h)$. The left continuity in t is similar, by writing the interval $[0, t - h]$ as the difference between $[0, t]$ and $(t - h, t]$ for $h > 0$. For the continuity in x , similarly as above, we see that $E|u_{n+1}(t, x + h) - u_{n+1}(t, x)|^p$ is bounded by:

$$D_{\sigma}^p 2^{p-1} (1 + K_n(t)) \int_0^t \int_{\mathcal{O}} |G(s, x + h, y) - G(s, x, y)|^p dy ds,$$

which converges to 0 as $|h| \rightarrow 0$ due to (51). This finishes the proof of (iv).

We denote $M_n(t) = \sup_{x \in \mathcal{O}} E|u_n(t, x)|^p$. Similarly to (58), we have:

$$M_n(t) \leq C_1 \int_0^t (1 + M_{n-1}(s)) J_p(t - s) ds, \quad \forall n \geq 1,$$

where $C_1 = C_{\alpha, p} K^{p-\alpha} D_{\sigma}^p 2^{p-1}$. By applying Lemma 15 of Erratum to [6] with $f_n = M_n$, $k_1 = 0$, $k_2 = 1$ and $g(s) = C J_p(s)$, we obtain that:

$$\sup_{n \geq 0} \sup_{t \in [0, T]} M_n(t) < \infty \quad \text{for all } T > 0. \quad (59)$$

We now prove that $\{u_n(t, x)\}_n$ converges in $L^p(\Omega)$, uniformly in $(t, x) \in [0, T] \times \mathcal{O}$. To see this, let $U_n(t) = \sup_{x \in \mathcal{O}} E|u_{n+1}(t, x) - u_n(t, x)|^p$ for $n \geq 0$. Using the moment inequality (45) (or (49)) and (53), we have:

$$U_n(t) \leq C_2 \int_0^t U_{n-1}(s) J_p(t - s) ds$$

where $C_2 = C_{\alpha,p} K^{p-\alpha} C_\sigma^p$. By Lemma 15 of Erratum to [6], $\sum_{n \geq 0} U_n(t)^{1/p}$ converges uniformly on $[0, T]$. (Note that this lemma is valid for all $p > 0$.)

We denote by $u(t, x)$ the limit of $u_n(t, x)$ in $L^p(\Omega)$. One can show that u satisfies properties (ii)-(iv) listed above. So u has a predictable modification. This modification is a solution of (56). To prove uniqueness, let v be another solution and denote $H(t) = \sup_{x \in \mathcal{O}} E|u(t, x) - v(t, x)|^p$. Then

$$H(t) \leq C_2 \int_0^t H(s) J_p(t-s) ds.$$

Using (52), it follows that $H(t) = 0$ for all $t > 0$. \square

6.2 Proof of Theorem 6.1: case $\alpha < 1$

In this case, for any $t > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$, we have: (see (11))

$$Z(t, B) = \int_{[0,t] \times B \times (\mathbb{R} \setminus \{0\})} z N(ds, dx, dz).$$

The characteristic function of $Z(t, B)$ is given by:

$$E(e^{iuZ(t,B)}) = \exp \left\{ t|B| \int_{\mathbb{R} \setminus \{0\}} (e^{iuz} - 1) \nu_\alpha(dz) \right\}, \quad \forall u \in \mathbb{R}.$$

Note that $\{Z(t, B)\}_{t \geq 0}$ is *not* a compound Poisson process since ν_α is infinite.

We introduce the stopping times $(\tau_K)_{K \geq 1}$, as on page 239 of [21]:

$$\tau_K(B) = \inf\{t > 0; |Z(t, B) - Z(t-, B)| > K\},$$

where $Z(t-, B) = \lim_{s \uparrow t} Z(s, B)$. Clearly, $\tau_L(B) \geq \tau_K(B)$ for all $L > K$.

We first investigate the relationship between Z and Z_K and the properties of $\tau_K(B)$. Using construction (55) of N and definition (39) of Z_K , we have:

$$\begin{aligned} Z(t, B) &= \sum_{i,j,k \geq 1} Z_i^{j,k} 1_{\{T_i^{j,k} \leq t\}} 1_{\{X_i^{j,k} \in B\}} =: \sum_{j,k \geq 1} Z^{j,k}(t, B) \\ Z_K(t, B) &= \sum_{i,j,k \geq 1} Z_i^{j,k} 1_{\{|Z_i^{j,k}| \leq K\}} 1_{\{T_i^{j,k} \leq t\}} 1_{\{X_i^{j,k} \in B\}}. \end{aligned}$$

We observe that $\{Z^{j,k}(t, B)\}_{t \geq 0}$ is a compound Poisson process with

$$E(e^{iuZ^{j,k}(t,B)}) = \exp \left\{ t|\mathcal{O}_k \cap B| \int_{U_j} (e^{iuz} - 1) \nu_\alpha(dz) \right\} \quad \forall u \in \mathbb{R}.$$

Note that $\tau_K(B) > T$ means that all the jumps of $\{Z(t, B)\}_{t \geq 0}$ in $[0, T]$ are smaller than K in modulus, i.e. $\{\tau_K(B) > T\} = \{\omega; |Z_i^{j,k}(\omega)| \leq K \text{ for all } i, j, k \geq 1 \text{ for which } T_i^{j,k}(\omega) \leq T \text{ and } X_i^{j,k}(\omega) \in B\}$. Hence, on $\{\tau_K(B) > T\}$,

$$Z([0, t] \times A) = Z_K([0, t] \times A) = Z_L([0, t] \times A),$$

for any $L > K$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ with $A \subset B$. Using an approximation argument and the construction of the integrals $I(X)$ and $I_K(X)$, it follows that for any $X \in \mathcal{L}_\alpha$ and for any $L > K$, a.s. on $\{\tau_K(B) > T\}$, we have:

$$I(X)(T, B) = I_K(X)(T, B) = I_L(X)(T, B). \quad (60)$$

The next result gives the probability of the event $\{\tau_K(B) > T\}$.

Lemma 6.6 *For any $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$,*

$$P(\tau_K(B) > T) = \exp(-T|B|K^{-\alpha}).$$

Consequently, $\lim_{K \rightarrow \infty} P(\tau_K(B) > T) = 1$ and $\lim_{K \rightarrow \infty} \tau_K(B) = \infty$ a.s.

Proof: Note that $\{\tau_K(B) > T\} = \bigcap_{j,k \geq 1} \{\tau_K^{j,k}(B) > T\}$, where

$$\tau_K^{j,k}(B) = \inf\{t > 0; |Z^{j,k}(t, B) - Z^{j,k}(t-, B)| > K\}$$

Since $\nu_\alpha(\{z; |z| > K\}) = K^{-\alpha}$ and $(\tau_K^{j,k}(B))_{j,k \geq 1}$ are independent, it is enough to prove that for any $j, k \geq 1$,

$$P(\tau_K^{j,k}(B) > T) = \exp\{-T|B \cap \mathcal{O}_k| \nu_\alpha(\{z; |z| > K\} \cap U_j)\}. \quad (61)$$

Note that $\{\tau_K^{j,k}(B) > T\} = \{\omega; |Z_i^{j,k}(\omega)| \leq K \text{ for all } i \text{ for which } T_i^{j,k} \leq T \text{ and } X_i^{j,k} \in B\}$ and $(T_n^{j,k})_{n \geq 1}$ are the jump times of a Poisson process with intensity $\lambda_{j,k}$. Hence

$$\begin{aligned} P(\tau_K^{j,k}(B) > T) &= \sum_{n \geq 0} \sum_{m=0}^n \sum_{I \subset \{1, \dots, n\}, \text{card}(I)=m} P(T_n^{j,k} \leq T < T_{n+1}^{j,k}) P\left(\bigcap_{i \in I} \{X_i^{j,k} \in B\}\right) \\ &\quad P\left(\bigcap_{i \in I} \{|Z_i^{j,k}| \leq K\}\right) P\left(\bigcap_{i \in I^c} \{X_i^{j,k} \notin B\}\right) \\ &= \sum_{n \geq 0} e^{-\lambda_{j,k}T} \frac{(\lambda_{j,k}T)^n}{n!} [1 - P(X_1^{j,k} \in B) P(|Z_1^{j,k}| > K)]^n \\ &= \exp\left\{-\lambda_{j,k}T P(X_1^{j,k} \in B) P(|Z_1^{j,k}| > K)\right\}, \end{aligned}$$

which yields (61).

To prove the last statement, let $A_K^{(n)} = \{\tau_K(B) > n\}$. Then $P(\overline{\lim}_K A_K^{(n)}) \geq \overline{\lim}_K P(A_K^{(n)}) = 1$ for any $n \geq 1$, and hence $P(\bigcap_{n \geq 1} \overline{\lim}_K A_K^{(n)}) = 1$. Hence, with probability 1, for any n , there exists some K_n such that $\tau_{K_n} > n$. Since $(\tau_K)_K$ is non-decreasing, this proves that $\tau_K \rightarrow \infty$ with probability 1. \square

Remark 6.7 The construction of $\tau_K(B)$ given above is due to [21] (in the case of a symmetric measure ν_α). This construction relies on the fact that B is a bounded set. Since $Z(t, \mathbb{R}^d)$ (and consequently $\tau_K(\mathbb{R}^d)$) is not well-defined, I could not see why this construction can also be used when $B = \mathbb{R}^d$, as it is claimed in [21]. To avoid this difficulty, one could try to use an increasing sequence $(E_n)_n$ of sets in $\mathcal{B}_b(\mathbb{R}^d)$ with $\bigcup_n E_n = \mathbb{R}^d$. Using (60) with $B = E_n$ and letting $n \rightarrow \infty$, we obtain that $I(X)(t, \mathbb{R}^d) = I_K(t, \mathbb{R}^d)$ a.s. on $\{t \leq \tau_K\}$, where $\tau_K = \inf_{n \geq 1} \tau_K(E_n)$. But $P(\tau_K > t) \leq P(\varliminf_n \{\tau_K(E_n) > t\}) \leq \varliminf_n P(\tau_K(E_n) > t) = \lim_n \exp(-t|E_n|K^{-\alpha}) = 0$ for any $t > 0$, which means that $\tau_K = 0$ a.s. Finding a suitable sequence $(\tau_K)_K$ of stopping times which could be used in the case $\mathcal{O} = \mathbb{R}^d$ remains an open problem.

In what follows, we denote $\tau_K = \tau_K(\mathcal{O})$. Let u_K be the solution of equation (56), whose existence is guaranteed by Theorem 6.5.

Lemma 6.8 *Under the assumptions of Theorem 6.1, for any $t > 0$, $x \in \mathcal{O}$ and $L > K$,*

$$u_K(t, x) = u_L(t, x) \quad \text{a.s. on } \{t \leq \tau_K\}.$$

Proof: By the definition of u_L and (60),

$$\begin{aligned} u_L(t, x) &= \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u_L(s, y)) Z_L(ds, dy) \\ &= \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u_L(s, y)) Z_K(ds, dy) \end{aligned}$$

a.s. on the event $\{t \leq \tau_K\}$. Using the definition of u_K and Proposition C.1 (Appendix C), we obtain that, with probability 1,

$$\begin{aligned} (u_K(t, x) - u_L(t, x))1_{\{t \leq \tau_K\}} &= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) (\sigma(u_K(s, y)) - \\ &\quad \sigma(u_L(s, y))) 1_{\{s \leq \tau_K\}} Z_K(ds, dy). \end{aligned}$$

Let $M(t) = \sup_{x \in \mathcal{O}} E(|u_K(t, x) - u_L(t, x)|^p 1_{\{t \leq \tau_K\}})$. Using the moment inequality (45) and the Lipschitz condition (53), we get:

$$M(t) \leq C \int_0^t J_p(t-s) M(s) ds,$$

where $C = C_{\alpha,p} K^{p-\alpha} C_\sigma^p$. Using (52), it follows that $M(t) = 0$ for all $t > 0$. \square

For any $t > 0, x \in \mathcal{O}$, let $\Omega_{t,x} = \bigcap_{L>K} \{t \leq \tau_K(t), u_K(t, x) \neq u_L(t, x)\}$, where L and K are positive integers. Let $\Omega_{t,x}^* = \Omega_{t,x} \cap \{\lim_{K \rightarrow \infty} \tau_K = \infty\}$. By Lemma 6.6 and Lemma 6.8, $P(\Omega_{t,x}^*) = 1$.

The next result concludes the proof of Theorem 6.1.

Proposition 6.9 *Under the assumptions of Theorem 6.1, the process $u = \{u(t, x); t \geq 0, x \in \mathcal{O}\}$ defined by:*

$$\begin{aligned} u(\omega, t, x) &= u_K(\omega, t, x) && \text{if } \omega \in \Omega_{t,x}^* \text{ and } t \leq \tau_K(\omega) \\ u(\omega, t, x) &= 0 && \text{if } \omega \notin \Omega_{t,x}^* \end{aligned}$$

is a mild solution of equation (1).

Proof: We first prove that u is predictable. Note that

$$u(t, x) = \lim_{K \rightarrow \infty} (u_K(t, x) 1_{\{t \leq \tau_K\}}) 1_{\Omega_{t,x}^*}.$$

The process $X(\omega, t, x) = 1_{\{t \leq \tau_K\}}(\omega)$ is clearly predictable, being in the class \mathcal{C} defined in Remark 4.1. By the definition of $\Omega_{t,x}$, since u_K, u_L are predictable, it follows that $(\omega, t, x) \mapsto 1_{\Omega_{t,x}^*}(\omega)$ is \mathcal{P} -measurable. Hence, u is predictable.

We now prove that u satisfies (2). Let $t > 0$ and $x \in \mathcal{O}$ be arbitrary. Using (60) and Proposition C.1 (Appendix C), with probability 1, we have:

$$\begin{aligned} 1_{\{t \leq \tau_K\}} u(t, x) &= 1_{\{t \leq \tau_K\}} u_K(t, x) \\ &= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u_K(s, y)) Z_K(ds, dy) \\ &= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u_K(s, y)) Z(ds, dy) \\ &= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u_K(s, y)) 1_{\{s \leq \tau_K\}} Z(ds, dy) \end{aligned}$$

$$\begin{aligned}
&= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u(s, y)) 1_{\{s \leq \tau_K\}} Z(ds, dy) \\
&= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u(s, y)) Z(ds, dy).
\end{aligned}$$

For the second last equality, we used the fact that processes $X(s, y) = 1_{[0, t]}(s)G(t-s, x, y)\sigma(u_K(s, y))1_{\{s \leq \tau_K\}}$ and $Y(s, y) = 1_{[0, t]}(s)G(t-s, x, y)\sigma(u(s, y))1_{\{s \leq \tau_K\}}$ are modifications of each other (i.e. $X(s, y) = Y(s, y)$ a.s. for all $s > 0, y \in \mathcal{O}$), and hence, $[X - Y]_{\alpha, t, \mathcal{O}} = 0$ and $I(X)(t, \mathcal{O}) = I(Y)(t, \mathcal{O})$ a.s. The conclusion follows letting $K \rightarrow \infty$, since $\tau_K \rightarrow \infty$ a.s. \square

6.3 Proof of Theorem 6.1: case $\alpha > 1$

In this case, for any $t > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$, we have: (see (12))

$$Z(t, B) = \int_{[0, t] \times B \times (\mathbb{R} \setminus \{0\})} z \widehat{N}(ds, dx, dz).$$

To introduce the stopping times $(\tau_K)_{K \geq 1}$ we use the same idea as in Section 9.7 of [22].

Let $M(t, B) = \sum_{j \geq 1} (L_j(t, B) - EL_j(t, B))$ and $P(t, B) = L_0(t, B)$, where $L_j(t, B) = L_j([0, t] \times B)$ was defined in Section 2. Note that $\{M(t, B)\}_{t \geq 0}$ is a zero-mean square-integrable martingale and $\{P(t, B)\}_{t \geq 0}$ is a compound Poisson process with $E[P(t, B)] = t|B|\mu$ where $\mu = \int_{|z| > 1} z \nu_\alpha(dz) = \beta \frac{\alpha}{\alpha-1}$. With this notation,

$$Z(t, B) = M(t, B) + P(t, B) - t|B|\mu. \quad (62)$$

We let $M_K(t, B) = P_K(t, B) - E[P_K(t, B)] = P_K(t, B) - t|B|\mu_K$, where

$$P_K(t, B) = \int_{[0, t] \times B \times (\mathbb{R} \setminus \{0\})} z 1_{\{1 < |z| \leq K\}} N(ds, dx, dz)$$

and $\mu_K = \int_{1 < |z| \leq K} z \nu_\alpha(dz)$. Recalling definition (40) of Z_K , it follows that:

$$Z_K(t, B) = M(t, B) + P_K(t, B) - t|B|\mu_K. \quad (63)$$

For any $K > 0$, we let

$$\tau_K(B) = \inf\{t > 0; |P(t, B) - P(t-, B)| > K\},$$

where $P(t-, B) = \lim_{s \uparrow t} P(s, B)$.

Lemma 6.6 holds again, but its proof is simpler than in the case $\alpha < 1$, since $\{P(t, B)\}_{t \geq 0}$ is a compound Poisson process. By (55),

$$\begin{aligned} P(t, B) &= \sum_{i,j,k \geq 1} Z_i^{j,k} 1_{\{|Z_i^{j,k}| > 1\}} 1_{\{T_i^{j,k} \leq t\}} 1_{\{X_i^{j,k} \in B\}} \\ P_K(t, B) &= \sum_{i,j,k \geq 1} Z_i^{j,k} 1_{\{1 < |Z_i^{j,k}| \leq K\}} 1_{\{T_i^{j,k} \leq t\}} 1_{\{X_i^{j,k} \in B\}}. \end{aligned}$$

Hence, on $\{\tau_K(B) > T\}$, for any $L > K$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ with $A \subset B$,

$$P([0, t] \times A) = P_K([0, t] \times A) = P_L([0, t] \times A).$$

Let $b_K = \mu - \mu_K = \int_{|z| > K} z \nu_\alpha(dz)$. Using (62) and (63), it follows that:

$$Z([0, t] \times A) = Z_K([0, t] \times A) - t|A|b_K = Z_L([0, t] \times A) - t|A|b_L$$

for any $L > K$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ with $A \subset B$. Let $p \in (\alpha, 2]$ be fixed. Using an approximation argument and the construction of the integrals $I(X)$ and $I_K(X)$, it follows that for any $X \in \mathcal{L}_\alpha$ and for any $L > K$, a.s. on $\{\tau_K(B) > T\}$, we have:

$$\begin{aligned} I(X)(T, B) &= I_K(X)(T, B) - b_K \int_0^T \int_{\mathcal{O}} X(s, y) dy ds \\ &= I_L(X)(T, B) - b_L \int_0^T \int_{\mathcal{O}} X(s, y) dy ds. \end{aligned} \quad (64)$$

We denote $\tau_K = \tau_K(\mathcal{O})$. We consider the following equation:

$$Lu(t, x) = \sigma(u(t, x)) \dot{Z}_K(t, x) - b_K \sigma(u(t, x)), \quad t > 0, x \in \mathcal{O} \quad (65)$$

with zero initial conditions and Dirichlet boundary conditions. A mild solution of (65) is a predictable process u which satisfies:

$$\begin{aligned} u(t, x) &= \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u(s, y)) Z_K(ds, dy) - \\ &\quad b_K \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u(s, y)) dy ds \quad \text{a.s.} \end{aligned}$$

for any $t > 0, x \in \mathcal{O}$. The existence and uniqueness of a mild solution of (65) can be proved similarly to Theorem 6.5. We omit these details. We denote this solution by v_K .

Lemma 6.10 *Under the assumptions of Theorem 6.1, for any $t > 0, x \in \mathcal{O}$ and $L > K$,*

$$v_K(t, x) = v_L(t, x) \quad \text{a.s. on } \{t \leq \tau_K\}.$$

Proof: By the definition of v_L and (64), a.s. on the event $\{t \leq \tau_K\}$, $v_L(t, x)$ is equal to

$$\begin{aligned} & \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(v_L(s, y)) Z_L(ds, dy) - b_L \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(v_L(s, y)) dy ds = \\ & \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(v_L(s, y)) Z_K(ds, dy) - b_K \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(v_L(s, y)) dy ds. \end{aligned}$$

Using the definition of v_K and Proposition C.1 (Appendix C), we obtain that, with probability 1,

$$\begin{aligned} (v_K(t, x) - v_L(t, x)) 1_{\{t \leq \tau_K\}} &= 1_{\{t \leq \tau_K\}} \left(\int_0^t \int_{\mathcal{O}} G(t-s, x, y) (\sigma(v_K(s, y)) - \sigma(v_L(s, y))) \right. \\ & \quad \left. 1_{\{s \leq \tau_K\}} Z_K(ds, dy) - \int_0^t \int_{\mathcal{O}} G(t-s, x, y) (\sigma(v_K(s, y)) - \sigma(v_L(s, y))) 1_{\{s \leq \tau_K\}} dy ds \right). \end{aligned}$$

Letting $M(t) = \sup_{x \in \mathcal{O}} E(|v_K(t, x) - v_L(t, x)|^p 1_{\{t \leq \tau_K\}})$, we see that $M(t) \leq 2^{p-1}(E|A(t, x)|^p + E|B(t, x)|^p)$ where

$$\begin{aligned} A(t, x) &= \int_0^t \int_{\mathcal{O}} G(t-s, x, y) (\sigma(v_K(s, y)) - \sigma(v_L(s, y))) 1_{\{s \leq \tau_K\}} Z_K(ds, dy) \\ B(t, x) &= \int_0^t \int_{\mathcal{O}} G(t-s, x, y) (\sigma(v_K(s, y)) - \sigma(v_L(s, y))) 1_{\{s \leq \tau_K\}} dy ds. \end{aligned}$$

We estimate separately the two terms. For the first term, we use the moment inequality (49) and the Lipschitz condition (53). We get:

$$\sup_{x \in \mathcal{O}} E|A(t, x)|^p \leq C \int_0^t J_p(t-s) M(s) ds, \quad (66)$$

where $C = C_{\alpha, p} K^{p-\alpha} C_{\sigma}^p$. For the second term, we use Hölder's inequality $|\int f g d\mu| \leq (\int |f|^p d\mu)^{1/p} (\int |g|^q d\mu)^{1/q}$ with $f(s, y) = G(t-s, x, y)^{1/p} (\sigma(v_K(s, y)) - \sigma(v_L(s, y))) 1_{\{s \leq \tau_K\}}$ and $g(s, y) = G(t-s, x, y)^{1/q}$, where $p^{-1} + q^{-1} = 1$. Hence,

$$|B(t, x)|^p \leq C_{\sigma}^p K_t^{p/q} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) |v_K(s, y) - v_L(s, y)|^p 1_{\{s \leq \tau_K\}} dy ds,$$

where $K_t = \int_0^t J_1(s)ds < \infty$. (Since \mathcal{O} is a bounded set, $J_1(s) \leq C J_p(s)^{1/p}$ where C is a constant depending on $|\mathcal{O}|$ and p . Since $p > 1$, $\int_0^t J_p(s)^{1/p} ds \leq c_t (\int_0^t J_p(s) ds)^{1/p} < \infty$ by (52). This shows that $K_t < \infty$.) Therefore,

$$\sup_{x \in \mathcal{O}} E|B(t, x)|^p \leq C_t \int_0^t J_1(t-s)M(s)ds, \quad (67)$$

where $C_t = C_\sigma^p K_t^{p/q}$. From (66) and (67), we obtain that:

$$M(t) \leq C'_t \int_0^t (J_p(t-s) + J_1(t-s))M(s)ds$$

where $C'_t = 2^{p-1}(C \vee C_t)$. This implies that $M(t) = 0$ for all $t > 0$. \square

For any $t > 0$ and $x \in \mathcal{O}$, we let $\Omega_{t,x} = \bigcap_{L>K} \{t \leq \tau_K, v_K(t, x) \neq v_L(t, x)\}$ where K and L are positive integers, and $\Omega_{t,x}^* = \Omega_{t,x} \cap \{\lim_{K \rightarrow \infty} \tau_K = \infty\}$. By Lemma 6.10, $P(\Omega_{t,x}^*) = 1$.

Proposition 6.11 *Under the assumptions of Theorem 6.1, the process $u = \{u(t, x); t \geq 0, x \in \mathcal{O}\}$ defined by:*

$$\begin{aligned} u(\omega, t, x) &= v_K(\omega, t, x) & \text{if } \omega \in \Omega_{t,x}^* \text{ and } t \leq \tau_K(\omega) \\ u(\omega, t, x) &= 0 & \text{if } \omega \notin \Omega_{t,x}^* \end{aligned}$$

is a mild solution of equation (1).

Proof: We proceed as in the proof of Proposition 6.9. In this case, with probability 1, we have:

$$\begin{aligned} 1_{\{t \leq \tau_K\}} u(t, x) &= 1_{\{t \leq \tau_K\}} \left(\int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u(s, y)) Z(ds, dy) - \right. \\ &\quad \left. b_K \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u(s, y)) dy ds \right). \end{aligned}$$

The conclusion follows letting $K \rightarrow \infty$, since $\tau_K \rightarrow \infty$ a.s. and $b_K \rightarrow 0$. \square

A Some auxiliary results

The following result is used in the proof of Theorem 4.3.

Lemma A.1 *If X has a $S_\alpha(\sigma, \beta, 0)$ distribution then*

$$\lambda^\alpha P(|X| > \lambda) \leq c_\alpha^* \sigma^\alpha \quad \text{for all } \lambda > 0,$$

where $c_\alpha^* > 0$ is a constant depending only on α .

Proof: *Step 1.* We first prove the result for $\sigma = 1$. We treat only the right tail, the left tail being similar. We denote X by X_β to emphasize the dependence on β . By Property 1.2.15 of [27], $\lim_{\lambda \rightarrow \infty} \lambda^\alpha P(X_\beta > \lambda) = C_\alpha \frac{1+\beta}{2}$, where $C_\alpha = (\int_0^\infty x^{-\alpha} \sin x dx)^{-1}$. We use the fact that for any $\beta \in [0, 1]$,

$$P(X_1 < \lambda) \leq P(X_\beta < \lambda) \leq P(X_0 < \lambda)$$

for all $\lambda \in \mathbb{R}$, $|\lambda| > \lambda_\alpha$ for some $\lambda_\alpha > 0$ (see Section 1.5 of [19] or p.37 of [27]). If $\beta \in [-1, 0]$, $-X_\beta$ has the same distribution as $X_{-\beta}$, and hence for $\lambda > \lambda_\alpha$,

$$P(X_\beta > \lambda) = P(X_{-\beta} < -\lambda) \leq P(X_0 < -\lambda) = P(X_0 > \lambda) \leq P(X_1 > \lambda).$$

Since $\lim_{\lambda \rightarrow \infty} \lambda^\alpha P(X_1 > \lambda) = C_\alpha$, there exists $\lambda_\alpha^* > \lambda_\alpha$ such that

$$\lambda^\alpha P(X_1 > \lambda) < 2C_\alpha \quad \text{for all } \lambda > \lambda_\alpha^*.$$

It follows that $\lambda^\alpha P(X_\beta > \lambda) < 2C_\alpha$ for all $\lambda > \lambda_\alpha^*$ and $\beta \in [-1, 1]$. Clearly, for all $\lambda \in (0, \lambda_\alpha^*]$ and $\beta \in [-1, 1]$, $\lambda^\alpha P(X_\beta > \lambda) \leq \lambda^\alpha \leq (\lambda_\alpha^*)^\alpha$.

Step 2. We now consider the general case. Since X/σ has a $S_\alpha(1, \beta, 0)$ distribution, by Step 1, it follows that $\lambda^\alpha P(|X| > \sigma\lambda) \leq c_\alpha^*$ for any $\lambda > 0$. The conclusion follows multiplying by σ^α . \square

In the proof of Theorem 4.3 and Lemma A.3 below, we use the following remark, due to Adam Jakubowski (personal communication).

Remark A.2 Let X be a random variable such that $P(|X| > \lambda) \leq K\lambda^{-\alpha}$ for all $\lambda > 0$, for some $K > 0$ and $\alpha \in (0, 2)$. Then, for any $A > 0$,

$$E(|X| 1_{\{|X| \leq A\}}) \leq \int_0^A P(|X| > t) dt \leq K \frac{1}{1-\alpha} A^{1-\alpha} \quad \text{if } \alpha < 1,$$

$$E(|X| 1_{\{|X| > A\}}) \leq \int_A^\infty P(|X| > t) dt + AP(|X| > A) \leq K \frac{\alpha}{\alpha-1} A^{1-\alpha} \quad \text{if } \alpha > 1,$$

$$E(X^2 1_{\{|X| \leq A\}}) \leq 2 \int_0^A tP(|X| > t) dt \leq K \frac{2}{2-\alpha} A^{2-\alpha} \quad \text{for any } \alpha \in (0, 2).$$

The next result is a generalization of Lemma 2.1 of [10] to the case of non-symmetric random variables. This result is used in the proof of Lemma 5.1 and Proposition 5.4.

Lemma A.3 *Let $(\eta_k)_{k \geq 1}$ be independent random variables such that*

$$\sup_{\lambda > 0} \lambda^\alpha P(|\eta_k| > \lambda) \leq K \quad \forall k \geq 1 \quad (68)$$

for some $K > 0$ and $\alpha \in (0, 2)$. If $\alpha > 1$, we assume that $E(\eta_k) = 0$ for all k , and if $\alpha = 1$, we assume that η_k has a symmetric distribution for all k . Then for any sequence $(a_k)_{k \geq 1}$ of real numbers, we have:

$$\sup_{\lambda > 0} \lambda^\alpha P\left(\left|\sum_{k \geq 1} a_k \eta_k\right| > \lambda\right) \leq r_\alpha K \sum_{k \geq 1} |a_k|^\alpha \quad (69)$$

where $r_\alpha > 0$ is a constant depending only on α .

Proof: We consider the intersection of the event on the left-hand side of (69) with the event $\{\sup_{k \geq 1} |a_k \eta_k| > \lambda\}$ and its complement. Hence,

$$P\left(\left|\sum_{k \geq 1} a_k \eta_k\right| > \lambda\right) \leq \sum_{k \geq 1} P(|a_k \eta_k| > \lambda) + P\left(\left|\sum_{k \geq 1} a_k \eta_k 1_{\{|a_k \eta_k| \leq \lambda\}}\right| > \lambda\right) =: I + II.$$

Using (68), we have $I \leq K \lambda^{-\alpha} \sum_{k \geq 1} |a_k|^\alpha$. To treat II , we consider 3 cases.

Case 1. $\alpha < 1$. By Markov's inequality and Remark A.2, we have:

$$II \leq \frac{1}{\lambda} \sum_{k \geq 1} |a_k| E(|\eta_k| 1_{\{|a_k \eta_k| \leq \lambda\}}) \leq K \frac{1}{1 - \alpha} \lambda^{-\alpha} \sum_{k \geq 1} |a_k|^\alpha.$$

Case 2. $\alpha > 1$. Let $X = \sum_{k \geq 1} a_k \eta_k 1_{\{|a_k \eta_k| \leq \lambda\}}$. Since $E(\sum_{k \geq 1} a_k \eta_k) = 0$,

$$|E(X)| = \left|E\left(\sum_{k \geq 1} a_k \eta_k 1_{\{|a_k \eta_k| > \lambda\}}\right)\right| \leq \sum_{k \geq 1} |a_k| E(|\eta_k| 1_{\{|a_k \eta_k| > \lambda\}}) \leq \frac{K\alpha}{\alpha - 1} \lambda^{1-\alpha} \sum_{k \geq 1} |a_k|^\alpha,$$

where we used Remark A.2 for the last inequality. From here, we infer that

$$|E(X)| < \frac{\lambda}{2} \quad \text{for any } \lambda > \lambda_\alpha,$$

where $\lambda_\alpha^\alpha = 2K \frac{\alpha}{\alpha-1} \sum_{k \geq 1} |a_k|^\alpha$. By Chebyshev's inequality, for any $\lambda > \lambda_\alpha$,

$$\begin{aligned} II &= P(|X| > \lambda) \leq P(|X - E(X)| > \lambda - |E(X)|) \leq \frac{4}{\lambda^2} E|X - E(X)|^2 \\ &\leq \frac{4}{\lambda^2} \sum_{k \geq 1} a_k^2 E(\eta_k^2 1_{\{|a_k \eta_k| \leq \lambda\}}) \leq \frac{8K}{2-\alpha} \lambda^{-\alpha} \sum_{k \geq 1} |a_k|^\alpha, \end{aligned}$$

using Remark A.2 for the last inequality. On the other hand, if $\lambda \in (0, \lambda_\alpha]$,

$$II = P(|X| > \lambda) \leq 1 \leq \lambda_\alpha^\alpha \lambda^{-\alpha} = 2K \frac{\alpha}{\alpha-1} \lambda^{-\alpha} \sum_{k \geq 1} |a_k|^\alpha.$$

Case 3. $\alpha = 1$. Since η_k has a symmetric distribution, we can use the original argument of [10]. \square

B Fractional power of the Laplacian

Let $\overline{G}(t, x)$ be the fundamental solution of $\frac{\partial u}{\partial t} + (-\Delta)^\gamma u = 0$ on \mathbb{R}^d , $\gamma > 0$.

Lemma B.1 *For any $p > 1$, there exist some constants $c_1, c_2 > 0$ depending on d, p, γ such that*

$$c_1 t^{-\frac{d}{2\gamma}(p-1)} \leq \int_{\mathbb{R}^d} \overline{G}(t, x)^p dx \leq c_2 t^{-\frac{d}{2\gamma}(p-1)}.$$

Proof: The upper bound is given by Lemma B.23 of [22]. For the lower bound, we use the scaling property of the functions $(g_{t,\gamma})_{t>0}$. We have:

$$\begin{aligned} \overline{G}(t, x) &= \int_0^\infty \frac{1}{(4\pi t^{1/\gamma} r)^{d/2}} \exp\left(-\frac{|x|^2}{4t^{1/\gamma} r}\right) g_{1,\gamma}(r) dr \\ &\geq \int_1^\infty \frac{1}{(4\pi t^{1/\gamma} r)^{d/2}} \exp\left(-\frac{|x|^2}{4t^{1/\gamma} r}\right) g_{1,\gamma}(r) dr \\ &\geq \frac{1}{(4\pi t^{1/\gamma})^{d/2}} \exp\left(-\frac{|x|^2}{4t^{1/\gamma}}\right) C_{d,\gamma} \quad \text{with} \quad C_{d,\gamma} := \int_1^\infty r^{-d/2} g_{1,\gamma}(r) dr < \infty, \end{aligned}$$

and hence

$$\int_{\mathbb{R}^d} \overline{G}(t, x)^p dx \geq c'_{d,\gamma,p} t^{-\frac{dp}{2\gamma}} \int_{\mathbb{R}^d} \exp\left(-\frac{p|x|^2}{4t^{1/\gamma}}\right) dx = c_{d,p,\gamma} t^{-\frac{d}{2\gamma}(p-1)}.$$

\square

C A local property of the integral

The following result is the analogue of Proposition 8.11 of [22].

Proposition C.1 *Let $T > 0$ and $\mathcal{O} \subset \mathbb{R}^d$ be a Borel set. Let $X = \{X(t, x); t \geq 0, x \in \mathbb{R}^d\}$ be a predictable process such that $X \in \mathcal{L}_\alpha$ if $\alpha < 1$, or $X \in \mathcal{L}_p$ for some $p \in (\alpha, 2]$ if $\alpha > 1$. If \mathcal{O} is unbounded, assume in addition that X satisfies (38) if $\alpha < 1$, or X satisfies (46) for some $p \in (\alpha, 2)$, if $\alpha > 1$. Suppose that there exists an event $A \in \mathcal{F}_T$ such that*

$$X(\omega, t, x) = 0 \quad \text{for all } \omega \in A, t \in [0, T], x \in \mathcal{O}. \quad (70)$$

Then for any $K > 0$, $I(X)(T, \mathcal{O}) = I_K(X)(T, \mathcal{O}) = 0$ a.s. on A .

Proof: We only prove the result for $I(X)$, the proof for $I_K(X)$ being the same. Moreover, we include only the argument for $\alpha < 1$; the case $\alpha > 1$ is similar. The idea is to reduce the argument to the case when X is a simple process, as in the proof Proposition of 8.11 of [22].

Step 1. We show that the proof can be reduced to the case of a bounded set \mathcal{O} . Let $X_n(t, x) = X(t, x)1_{\mathcal{O}_n}(x)$ where $\mathcal{O}_n = \mathcal{O} \cap E_n$ and $(E_n)_n$ is an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_n E_n = \mathbb{R}^d$. Then $X_n \in \mathcal{L}_\alpha$ satisfies (70). By the dominated convergence theorem,

$$E \int_0^T \int_{\mathcal{O}} |X_n(t, x) - X(t, x)|^\alpha \rightarrow 0.$$

By the construction of the integral, $I(X_{n_k})(T, \mathcal{O}) \rightarrow I(X)(T, \mathcal{O})$ a.s. for a subsequence $\{n_k\}$. It suffices to show that $I(X_n)(T, \mathcal{O}) = 0$ a.s. on A for all n . But $I(X_n)(T, \mathcal{O}) = I(X_n)(T, \mathcal{O}_n)$ and \mathcal{O}_n is bounded.

Step 2. We show that the proof can be reduced to the case of a bounded processes. For this, let $X_n(t, x) = X(t, x)1_{\{|X(t, x)| \leq n\}}$. Clearly, $X_n \in \mathcal{L}_\alpha$ is bounded and satisfies (70) for all n . By the dominated convergence theorem, $[X_n - X]_\alpha \rightarrow 0$, and hence $I(X_{n_k})(T, \mathcal{O}) \rightarrow I(X)(T, \mathcal{O})$ a.s. for a subsequence $\{n_k\}$. It suffices to show that $I(X_n)(T, \mathcal{O}) = 0$ a.s. on A for all n .

Step 3. We show that the proof can be reduced to the case of bounded continuous processes. Assume that $X \in \mathcal{L}_\alpha$ is bounded and satisfies (70). For any $t > 0$ and $x \in \mathbb{R}^d$, we define

$$X_n(t, x) = n^{d+1} \int_{(t-1/n) \vee 0}^t \int_{(x-1/n, x] \cap \mathcal{O}} X(s, y) dy ds,$$

where $(a, b] = \{y \in \mathbb{R}^d; a_i < y_i \leq b_i \text{ for all } i = 1, \dots, d\}$. Clearly, X_n is bounded and satisfies (70). We prove that $X_n \in \mathcal{L}_\alpha$. Since X_n is bounded, $[X_n]_\alpha < \infty$. To prove that X_n is predictable, we consider

$$F(t, x) = \int_0^t \int_{(0, x] \cap \mathcal{O}} X(s, y) dy ds.$$

Since X is predictable, it is progressively measurable, i.e. for any $t > 0$, the map $(\omega, s, x) \mapsto X(\omega, s, x)$ from $\Omega \times [0, t] \times \mathbb{R}^d$ to \mathbb{R} is $\mathcal{F}_t \times \mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{R}^d)$ -measurable. Hence, $F(t, \cdot)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$ -measurable for any $t > 0$. Since the map $t \mapsto F(\omega, t, x)$ is left-continuous for any $\omega \in \Omega, x \in \mathbb{R}^d$, it follows that F is predictable, being in the class \mathcal{C} defined in Remark 4.1. Hence, X_n is predictable, being a sum of 2^{d+1} terms involving F .

Since F is continuous in (t, x) , X_n is continuous in (t, x) . By Lebesgue differentiation theorem in \mathbb{R}^{d+1} , $X_n(\omega, t, x) \rightarrow X(\omega, t, x)$ for any $\omega \in \Omega, t > 0, x \in \mathcal{O}$. By the bounded convergence theorem, $[X_n - X]_\alpha \rightarrow 0$. Hence $I(X_{n_k})(T, \mathcal{O}) \rightarrow I(X)(T, \mathcal{O})$ a.s. for a subsequence $\{n_k\}$. It suffices to show that $I(X_n)(T, \mathcal{O}) = 0$ a.s. on A for all n .

Step 4. Assume that $X \in \mathcal{L}_\alpha$ is bounded, continuous and satisfies (70). Let $(U_j^{(n)})_{j=1, \dots, m_n}$ be a partition of \mathcal{O} in Borel sets with Lebesgue measure smaller than $1/n$. Let $x_j^n \in U_j^{(n)}$ be arbitrary. Define

$$X_n(t, x) = \sum_{k=0}^{n-1} \sum_{j=1}^{m_n} X\left(\frac{kT}{n}, x_j^n\right) 1_{\left(\frac{kT}{n}, \frac{(k+1)T}{n}\right]}(t) 1_{U_j^{(n)}}(x).$$

Since X is continuous in (t, x) , $X_n(t, x) \rightarrow X(t, x)$. By the bounded convergence theorem, $[X_n - X]_\alpha \rightarrow 0$, and hence $I(X_{n_k})(T, \mathcal{O}) \rightarrow I(X)(T, \mathcal{O})$ a.s. for a subsequence $\{n_k\}$. Since on the event A ,

$$I(X_n)(T, \mathcal{O}) = \sum_{k=0}^{n-1} \sum_{j=1}^{m_n} X\left(\frac{kT}{n}, x_j^n\right) Z\left(\left(\frac{kT}{n}, \frac{(k+1)T}{n}\right] \times U_j^{(n)}\right) = 0,$$

it follows that $I(X)(T, \mathcal{O}) = 0$ a.s. on A . \square

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